Beyond Hoeffding and Chernoff

Trading conclusiveness for advantages in quantum hypothesis testing

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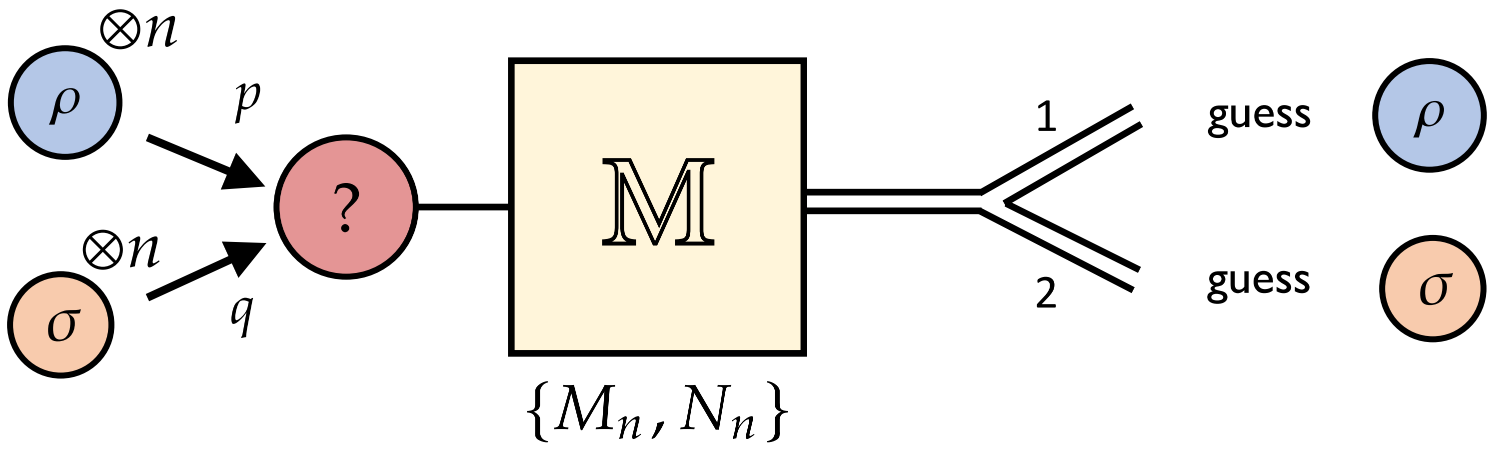
Quantum hypothesis testing



$$\alpha_n = \operatorname{Tr} N_n \rho^{\otimes n}$$
type I error

$$\beta_n = \operatorname{Tr} M_n \sigma^{\otimes n}$$
type II error

asymmetric hypothesis testing (Hoeffding)

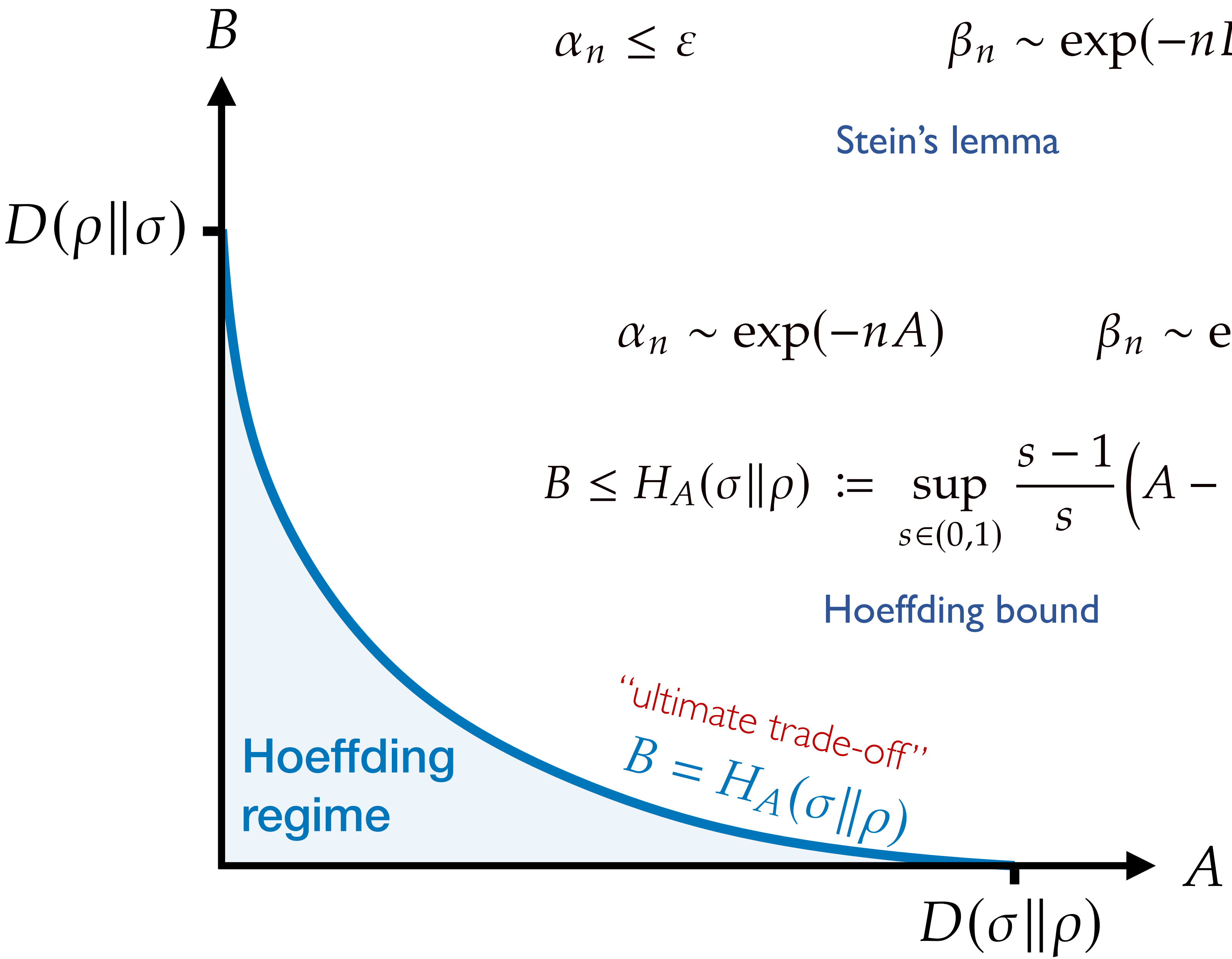
fix α_n, optimise β_n

understand the trade-offs between errors

symmetric hypothesis testing (Chernoff)

$$\mathrm{err}_n = p\,\alpha_n + q\,\beta_n$$

Exponent trade-offs



$$\alpha_n \le \varepsilon \qquad \beta_n \sim \exp(-nD(\rho\|\sigma))$$

Stein's lemma

$D(\rho\|\sigma) := \operatorname{Tr}\rho(\log\rho - \log\sigma)$

$$\alpha_n \sim \exp(-nA) \qquad \beta_n \sim \exp(-nB)$$

$$B \le H_A(\sigma\|\rho) := \sup_{s\in(0,1)} \frac{s-1}{s}\Big(A - D_s(\sigma\|\rho)\Big)$$

(Audenaert et al., CMP 2008)
(Hayashi, PRA 2007)

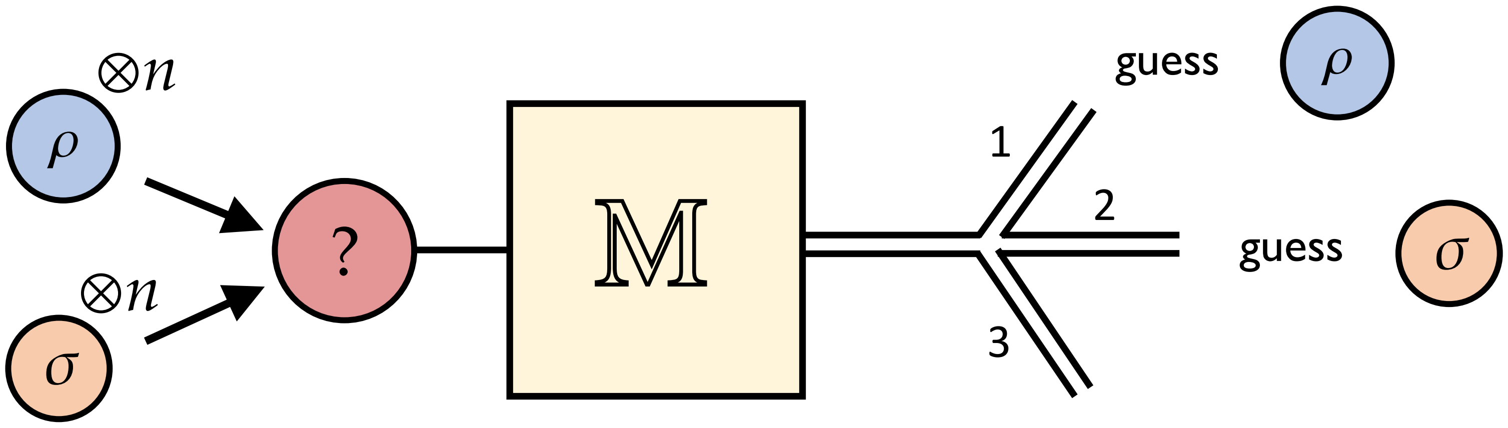
Hoeffding bound

$$D_s(\sigma\|\rho) = \frac{1}{s-1}\log\operatorname{Tr}\sigma^s\rho^{1-s}$$

Petz–Rényi divergence

What if we could say “I don’t know”?

Hypothesis testing with an inconclusive outcome



$$\{M_n, N_n, \mathbb{1} - M_n - N_n\}$$

$$\overline{\alpha}_n := \frac{\operatorname{Tr} N_n \rho^{\otimes n}}{\operatorname{Tr}(M_n + N_n)\rho^{\otimes n}}$$

conditional type I error

$$\overline{\beta}_n := \frac{\operatorname{Tr} M_n \sigma^{\otimes n}}{\operatorname{Tr}(M_n + N_n)\sigma^{\otimes n}}$$

conditional type II error

$$\pi_n(\rho) := \operatorname{Tr}(M_n + N_n)\rho^{\otimes n}$$

$$\pi_n(\sigma) := \operatorname{Tr}(M_n + N_n)\sigma^{\otimes n}$$

probability of conclusive discrimination

Classical intuition

classical probability distributions P Q

type (empirical distribution) $t_{x^n}(x) = \frac{1}{n} N(x|x^n)$

δ-typical set $\Pi^{\delta}_{P,n}$ sequences s.t. $\|t_{x^n} - P\|_\infty \le \delta$

$$\left(\Pi^{\delta}_{P,n},\ \Pi^{\delta}_{Q,n},\ \mathbb{1} - \Pi^{\delta}_{P,n} - \Pi^{\delta}_{Q,n}\right)$$

$$\overline{\alpha}_n \sim \exp(-nD(Q\|P)) \qquad \pi_n(P) \to 1$$

$$\overline{\beta}_n \sim \exp(-nD(P\|Q)) \qquad \pi_n(Q) \to 1$$

achieves both of the maximal exponents

arbitrarily high conclusiveness

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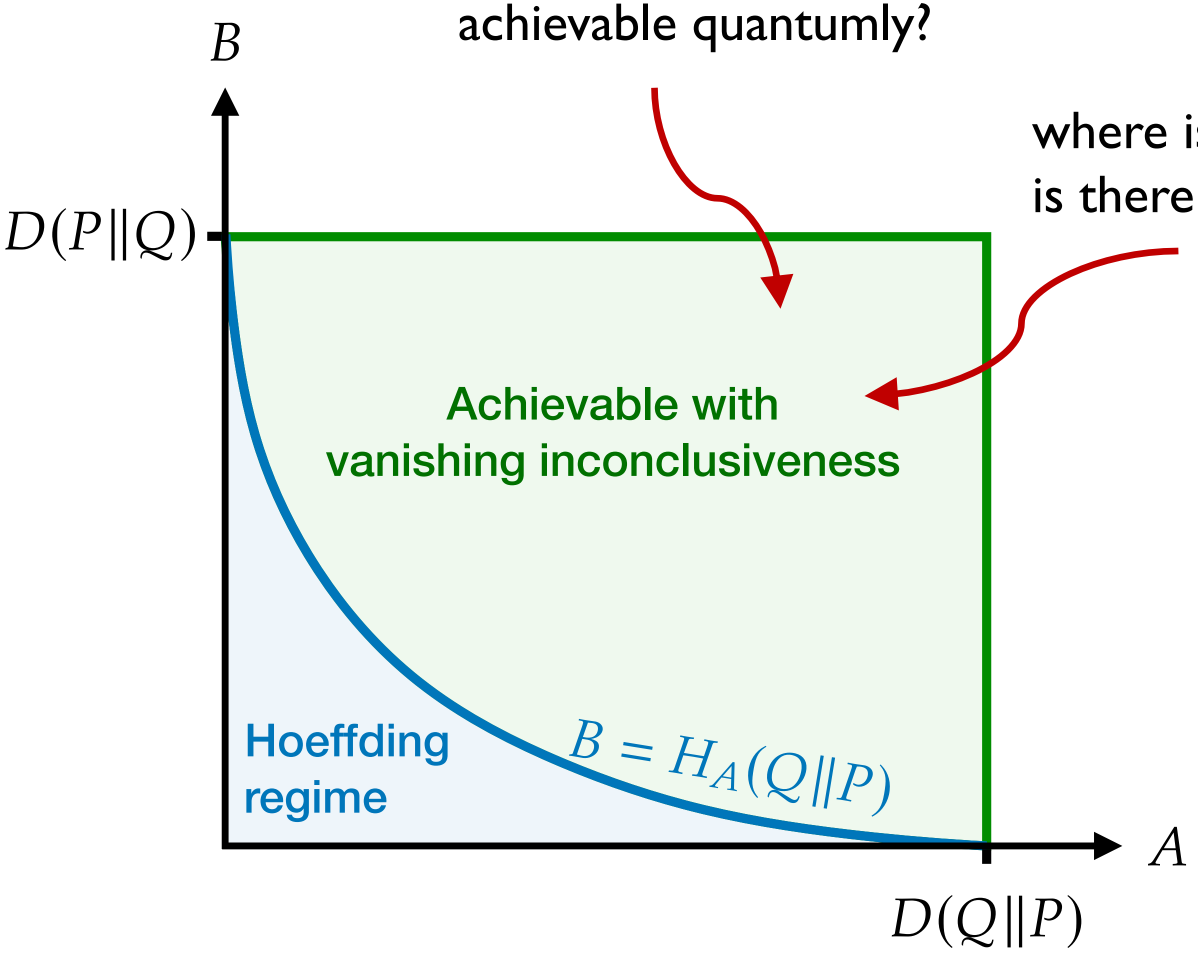
Classical intuition

achievable quantumly?

where is the catch?
is there some hidden cost?

what if one tries to achieve
even higher exponents?



Lifting to quantum?

$(\mathcal{M}_k)_k$ $$\frac{1}{k} D(\mathcal{M}_k(\rho^{\otimes k}) \| \mathcal{M}_k(\sigma^{\otimes k})) \to D(\rho\|\sigma)$$

measured relative entropy (Hiai and Petz, CMP 1991)

$(\mathcal{N}_k)_k$ $$\frac{1}{k} D(\mathcal{N}_k(\sigma^{\otimes k}) \| \mathcal{N}_k(\rho^{\otimes k})) \to D(\sigma\|\rho)$$

$$\overline{\alpha}_n \sim \exp(-nA) \qquad \qquad A \le \lim_{k\to\infty} \frac{1}{k} D(\mathcal{M}_k(\sigma^{\otimes k}) \| \mathcal{M}_k(\rho^{\otimes k}))$$

$$\Longleftrightarrow$$

$$\overline{\beta}_n \sim \exp(-nB) \qquad \qquad B \le \lim_{k\to\infty} \frac{1}{k} D(\mathcal{M}_k(\rho^{\otimes k}) \| \mathcal{M}_k(\sigma^{\otimes k}))$$

optimal for which relative entropy?

already beats Hoeffding, but highly unsatsifying

$$A \le D(\sigma\|\rho),\ B \le D^\star(\rho\|\sigma) \qquad = \qquad \lim_{\alpha\to 1} \widehat{D}_\alpha(\rho\|\sigma) = D(\rho\| \operatorname{diag}_\rho(\nu_1, \nu_2/\nu_1, \nu_3/\nu_2, \ldots, \nu_d/\nu_{d-1})),$$

$$\textit{with } \nu_k = \det(\sigma_{1:k,1:k}), \quad k = 1, \ldots, d.$$

$$A \le D^\star(\sigma\|\rho),\ B \le D(\rho\|\sigma)$$

(Audenaert and Datta, JMP 2015)

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Achievability from sequential hypothesis testing

choose measurement adaptively

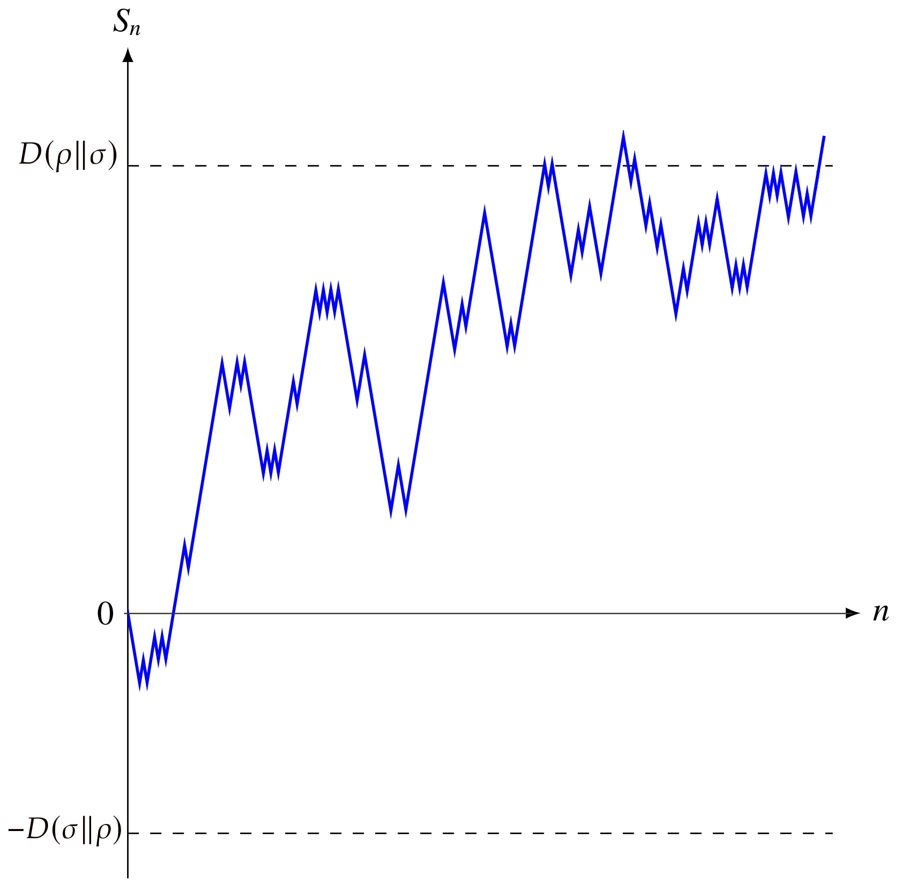
(Li, Tan, and Tomamichel, CMP 2022)

$$\overline{\alpha}_n \sim \exp(-nD(\sigma\|\rho))$$

$$\overline{\beta}_n \sim \exp(-nD(\rho\|\sigma))$$

$$\pi_n(\rho) \to 1$$

$$\pi_n(\sigma) \to 1$$



$$\overline{\alpha}_n \sim \exp(-nA) \quad \Longleftrightarrow \quad A \le D(\sigma\|\rho) \quad \Longleftrightarrow \quad A \le \lim_{k\to\infty} \frac{1}{k} D(\mathcal{M}_k(\sigma^{\otimes k})\|\mathcal{M}_k(\rho^{\otimes k}))$$

$$\overline{\beta}_n \sim \exp(-nB) \qquad B \le D(\rho\|\sigma) \qquad B \le \lim_{k\to\infty} \frac{1}{k} D(\mathcal{M}_k(\rho^{\otimes k})\|\mathcal{M}_k(\sigma^{\otimes k}))$$

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How to implement? Overhead costs?

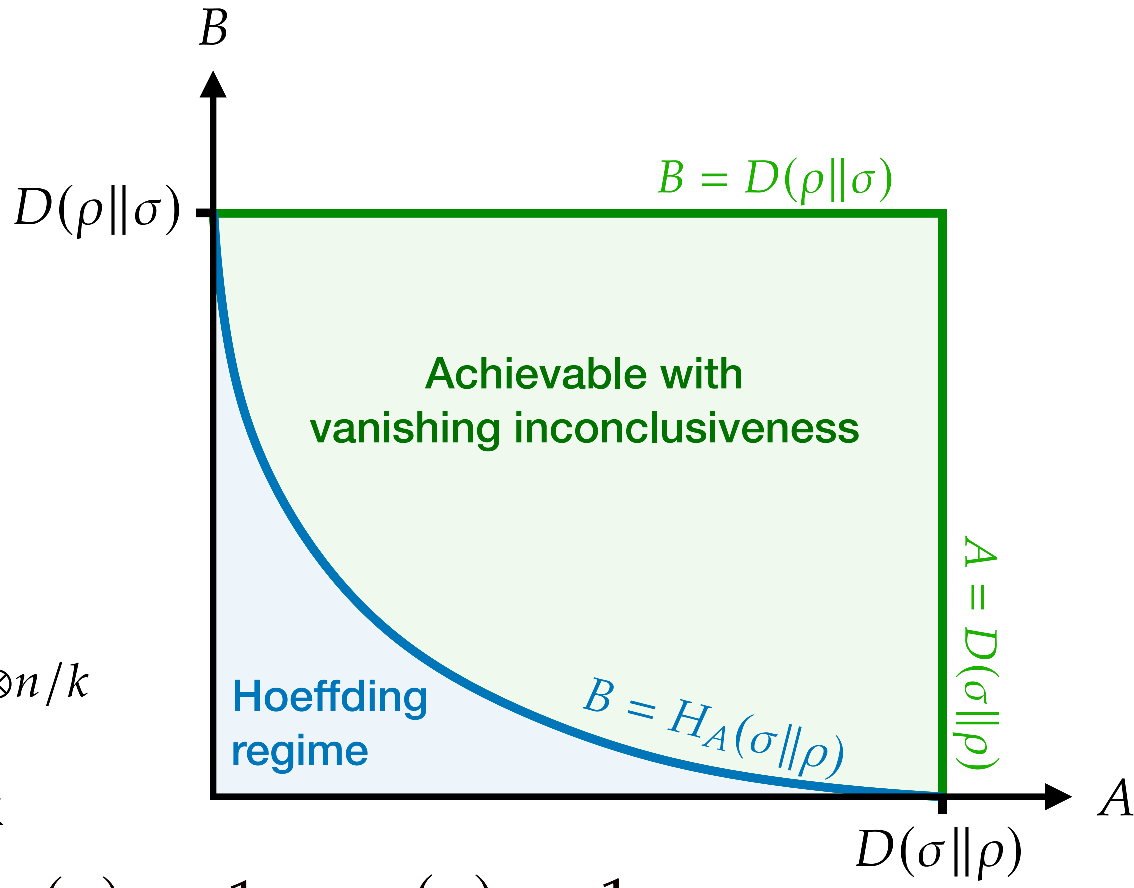
standard Hoeffding:

1. take n copies $\omega^{\otimes n}$
2. run a huge n-copy Neyman–Pearson test

$$\{\rho^{\otimes n} > \exp(\lambda)\, \sigma^{\otimes n}\}$$

3. asymptotically achievable exponents:

$$A\,, \qquad B \leq H_A(\sigma\|\rho)$$



allowing inconclusiveness:

1. take n copies, divide into blocks of k copies $\left(\omega^{\otimes k}\right)^{\otimes n/k}$
2. run an adaptive, 'sequential' protocol inside each block

2a. sometimes fails, but failure arbitrarily unlikely if $\pi_n(\rho) \to 1, \quad \pi_n(\sigma) \to 1$

3. achievable exponents: $A \leq D(\sigma\|\rho),\ B \leq D(\rho\|\sigma)$ as k grows

3a. a modest block size suffices: $D(\rho\|\sigma) = D_{\mathbb{M}}\left(\rho^{\otimes k}\|\sigma^{\otimes k}\right) + O\left(\frac{\log k}{k}\right)$

(Hiai and Petz, 1994)
(Hayashi, 2001)

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Achievable exponents

$$\overline{\alpha}_n \sim \exp(-nA)$$

$$\overline{\beta}_n \sim \exp(-nB)$$

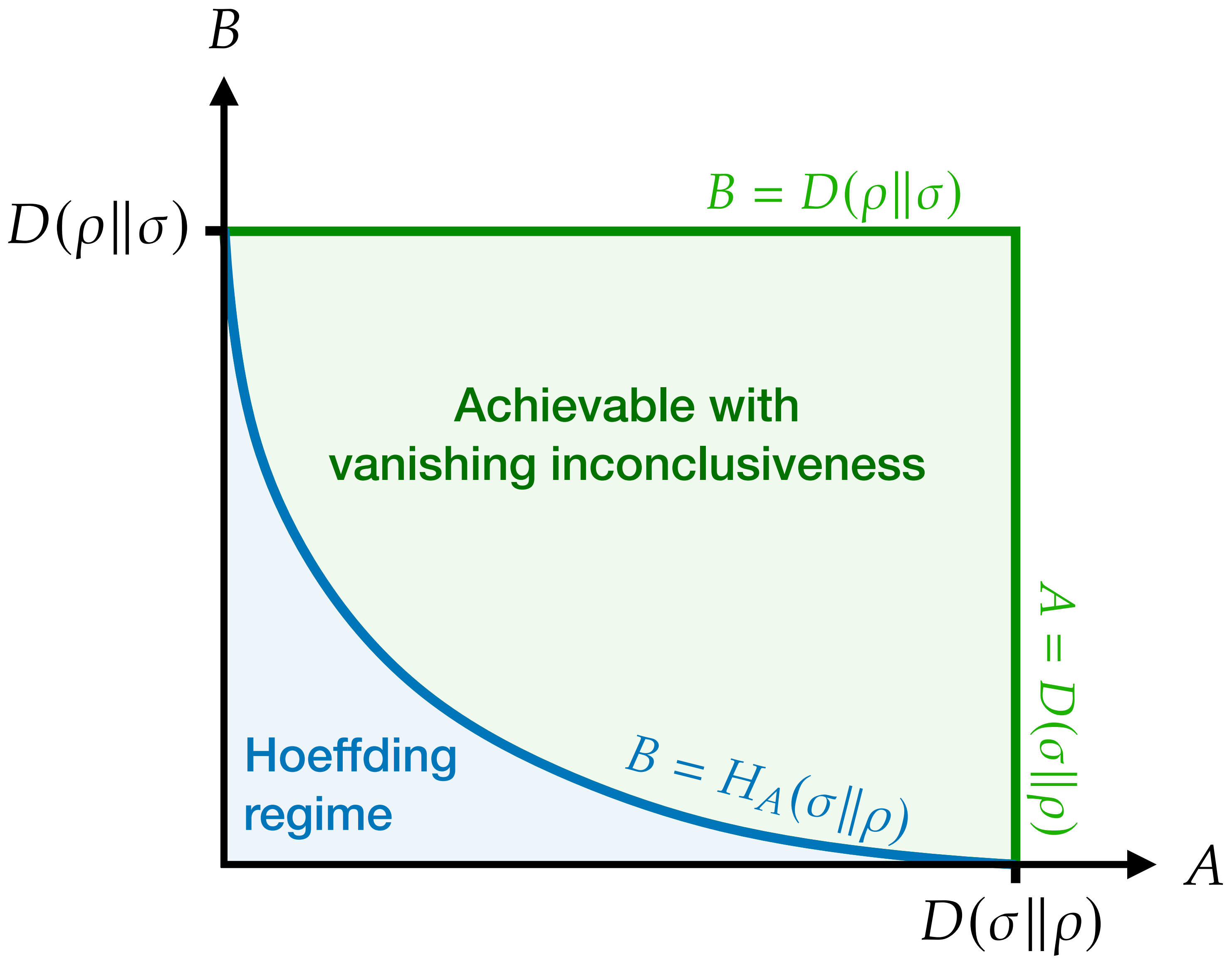
$$\pi_n(\rho) \to 1$$

$$\pi_n(\sigma) \to 1$$

possible to significantly exceed Hoeffding's bound with

arbitrarily high conclusiveness
= arbitrarily small overhead

+ easy implementation



Going further beyond: one-sided constraints

$$\overline{\alpha}_n \sim \exp(-nA) \qquad \overline{\beta}_n \sim \exp(-nB)$$

$$\pi_n(\rho) \to 1$$

achievability and converse

$$B \leq D(\rho\|\sigma) - H_A^*(\sigma\|\rho)$$

$$H_A^*(\sigma\|\rho) = \sup_{s>1} \frac{s-1}{s}\left(A - \widetilde{D}_s(\sigma\|\rho)\right)$$

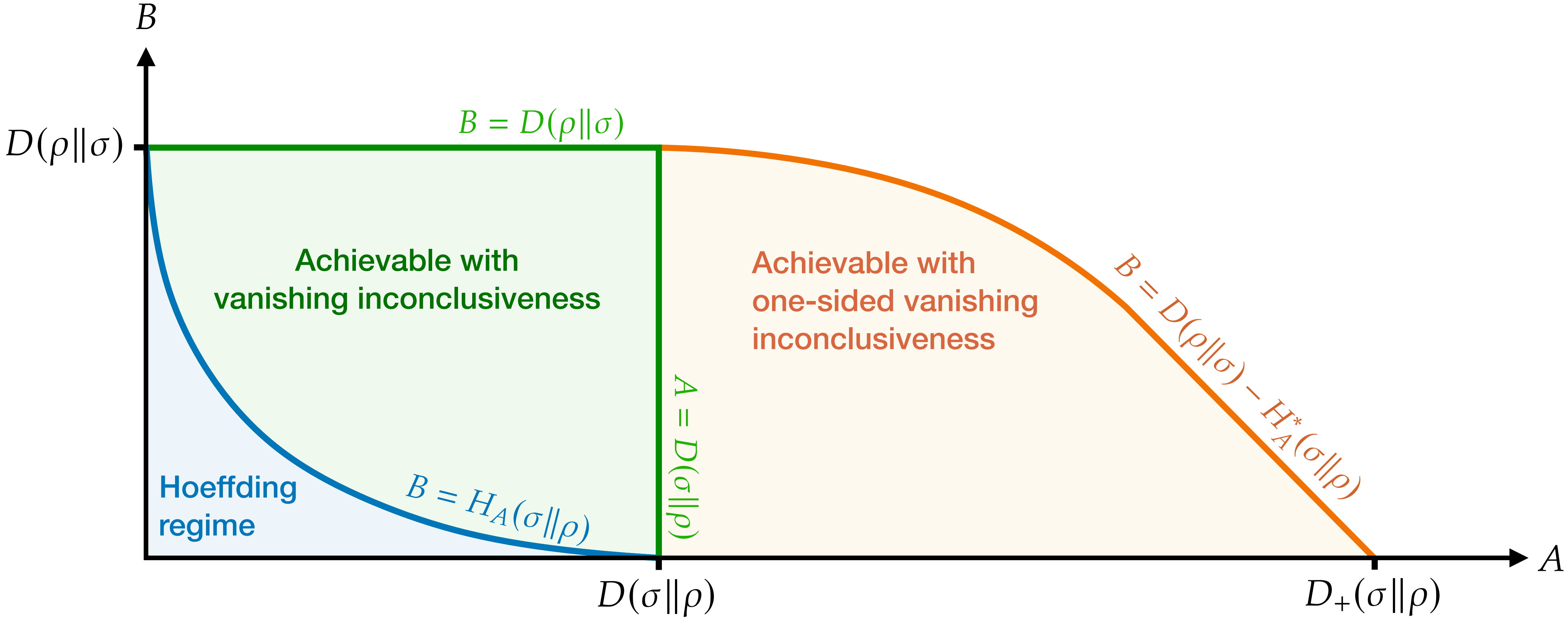
Hoeffding anti-divergence

(Mosonyi and Ogawa, 2015)

$$\widetilde{D}_s(\sigma\|\rho) = \frac{1}{s-1} \log \mathrm{Tr}\left(\rho^{\frac{1-s}{2s}} \sigma \rho^{\frac{1-s}{2s}}\right)^s$$

Going further beyond: one-sided constraints

$\overline{\alpha}_n \sim \exp(-nA)$ $\qquad$ $\overline{\beta}_n \sim \exp(-nB)$



$$D_+(\sigma\|\rho) = D_{\max}(\sigma\|\rho) + D(\rho\|\sigma)$$

Going further beyond: one-sided constraints

$$\overline{\alpha}_n \sim \exp(-nA) \qquad \overline{\beta}_n \sim \exp(-nB)$$

$$\pi_n(\rho) \to 1$$

exponents of distillation ⇔ exponents of hypothesis testing

(generalised quantum Stein's lemma)

(Brandão and Plenio, 2010)

But does it make sense? Actually, yeah

Quantum resource distillation $\rho^{\otimes n} \to |\psi\rangle\langle\psi|^{\otimes nR}$

$$\mathcal{E}_n(\rho^{\otimes n}) = \mathrm{Tr}(M_n \rho^{\otimes n})\, |\psi\rangle\langle\psi|^{\otimes nR} + \mathrm{Tr}\left[(\mathbb{1} - M_n)\rho^{\otimes n}\right] \tau_n$$

free-operation $\mathcal{E}_n(\sigma^{\otimes n}) = \sigma^{\otimes nR}$ "resource non-generating"

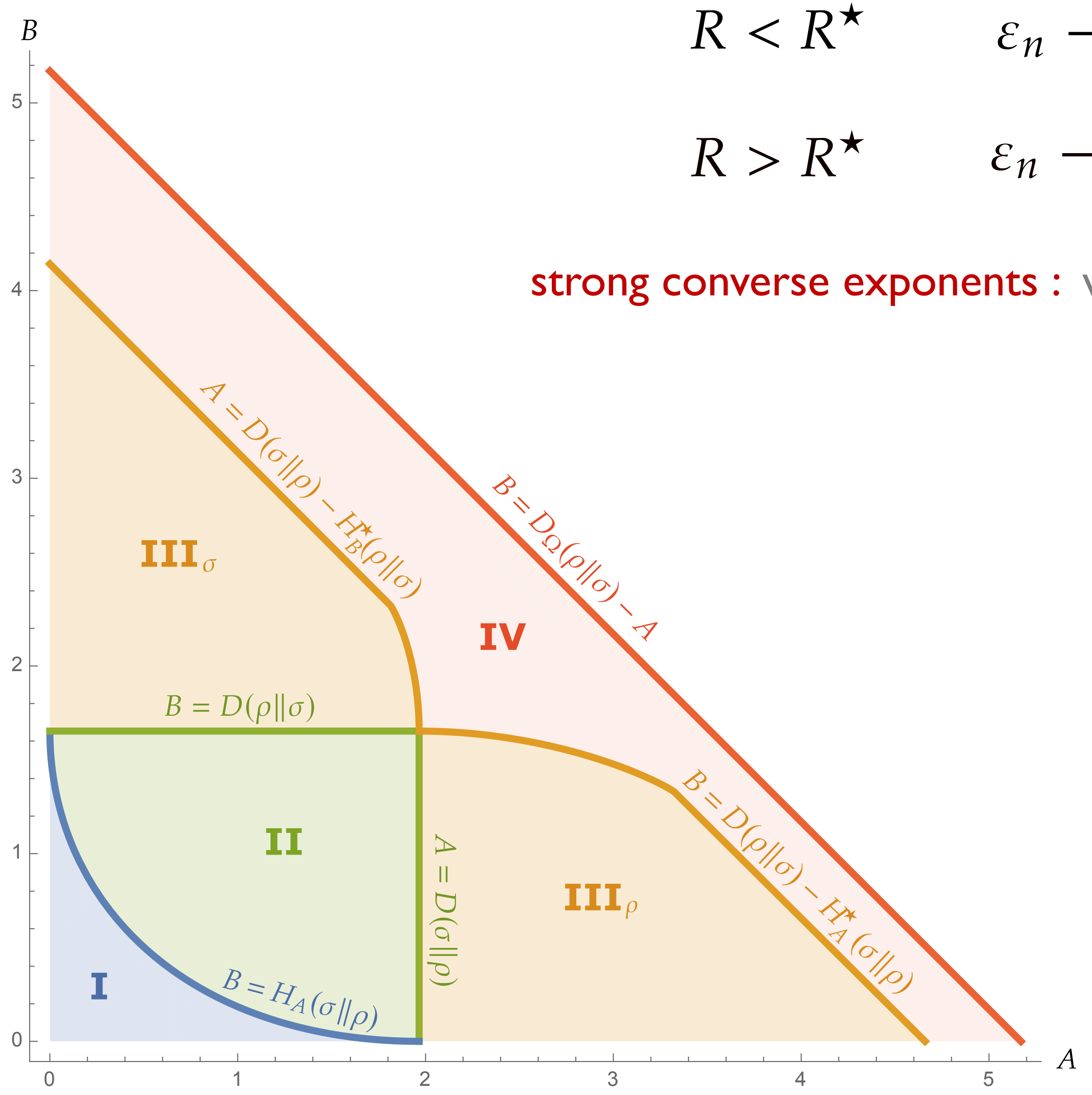
Do it probabilistically! $\mathcal{E}'_n(\rho^{\otimes n}) = \mathrm{Tr}(M_n \rho^{\otimes n})\, |\psi\rangle\langle\psi|^{\otimes nR} + \mathrm{Tr}(N_n \rho^{\otimes n}) \tau_n$

$$\rho^{\otimes n} \to \frac{\mathcal{E}'_n(\rho^{\otimes n})}{\pi_n(\rho)} \qquad \pi_n(\rho) \to 1$$

high probability of success

free-operation $\mathcal{E}'_n(\sigma^{\otimes n}) \propto \sigma^{\otimes nR}$

"Strong converse": exponents of conclusiveness



$R < R^\star \qquad \varepsilon_n \to 0$

$R > R^\star \qquad \varepsilon_n \to 1 \qquad 1 - \varepsilon_n \sim \exp(-nS)$

strong converse exponents : what is the penalty for exceeding the optimal rate?

$\pi_n(\rho) \to 0$

$\pi_n(\sigma) \to 0$

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Full range of trade-offs

$$\overline{\alpha}_n \sim \exp(-nA) \qquad \overline{\beta}_n \sim \exp(-nB)$$

$$\pi_n(\rho) \sim \exp(-nK) \qquad \pi_n(\sigma) \sim \exp(-nL)$$

achievability and converse

$$A + K \leq \inf_{t>1} \left(\frac{t}{t-1} L + \widetilde{D}_t(\sigma \| \rho) \right),$$

$$B + L \leq \inf_{s>1} \left(\frac{s}{s-1} K + \widetilde{D}_s(\rho \| \sigma) \right)$$

complete understanding of trade-offs
for all quantum states

new applications of sandwiched Rényi divergence

various insights follow:

what if A, B fixed, what is the best K?

$$K \geq \sup_{s,t>1} \left(\frac{s}{s-1} - \frac{t-1}{t} \right)^{-1} \left(B - \widetilde{D}_s(\rho \| \sigma) + \frac{t-1}{t} \left(A - \widetilde{D}_t(\sigma \| \rho) \right) \right)$$

what if K, L not constrained at all?

$$A + B \leq \widetilde{D}_\infty(\rho \| \sigma) + \widetilde{D}_\infty(\sigma \| \rho)$$

postselected hypothesis testing

(Regula, Lami, Wilde, 2024)

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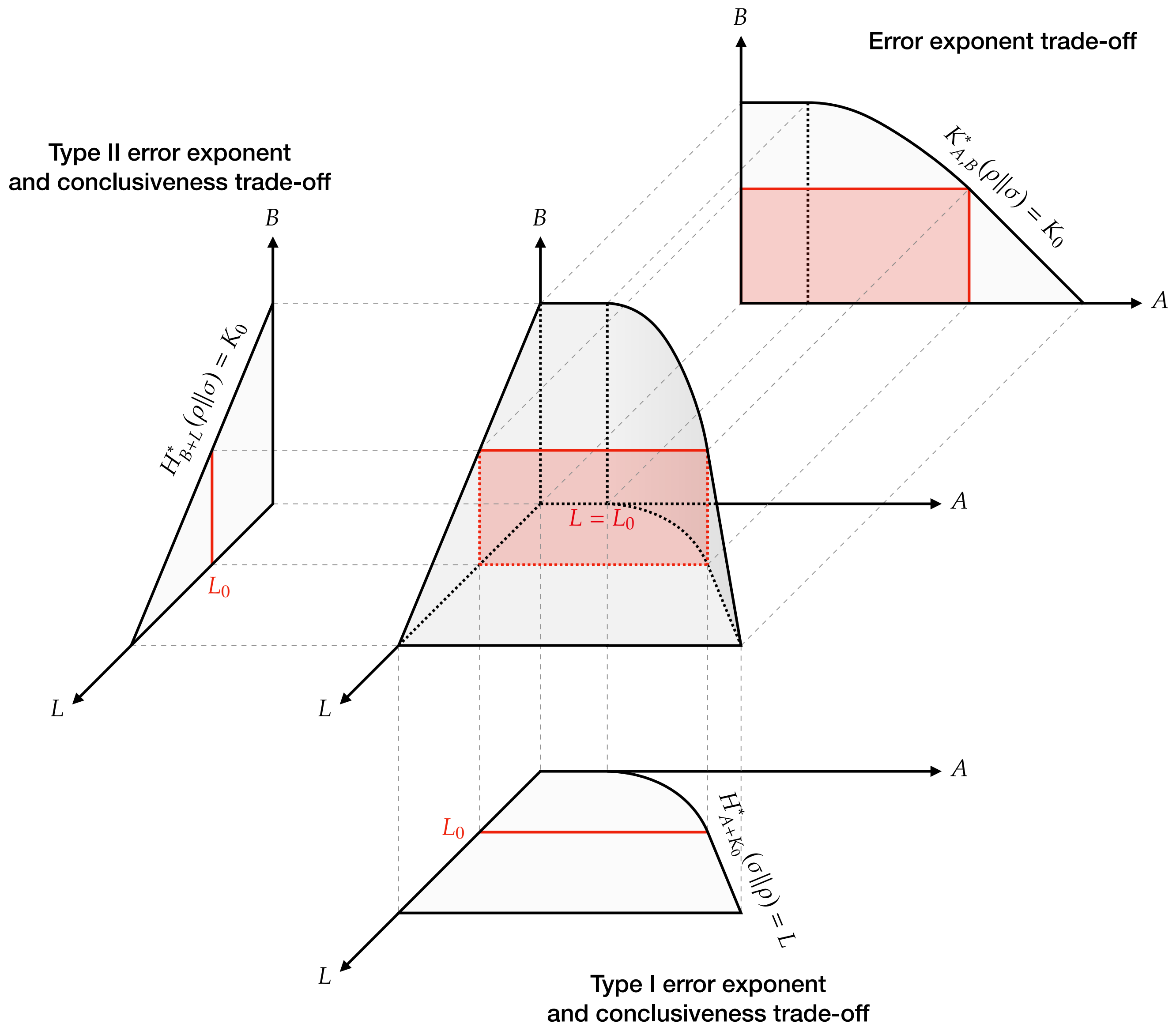
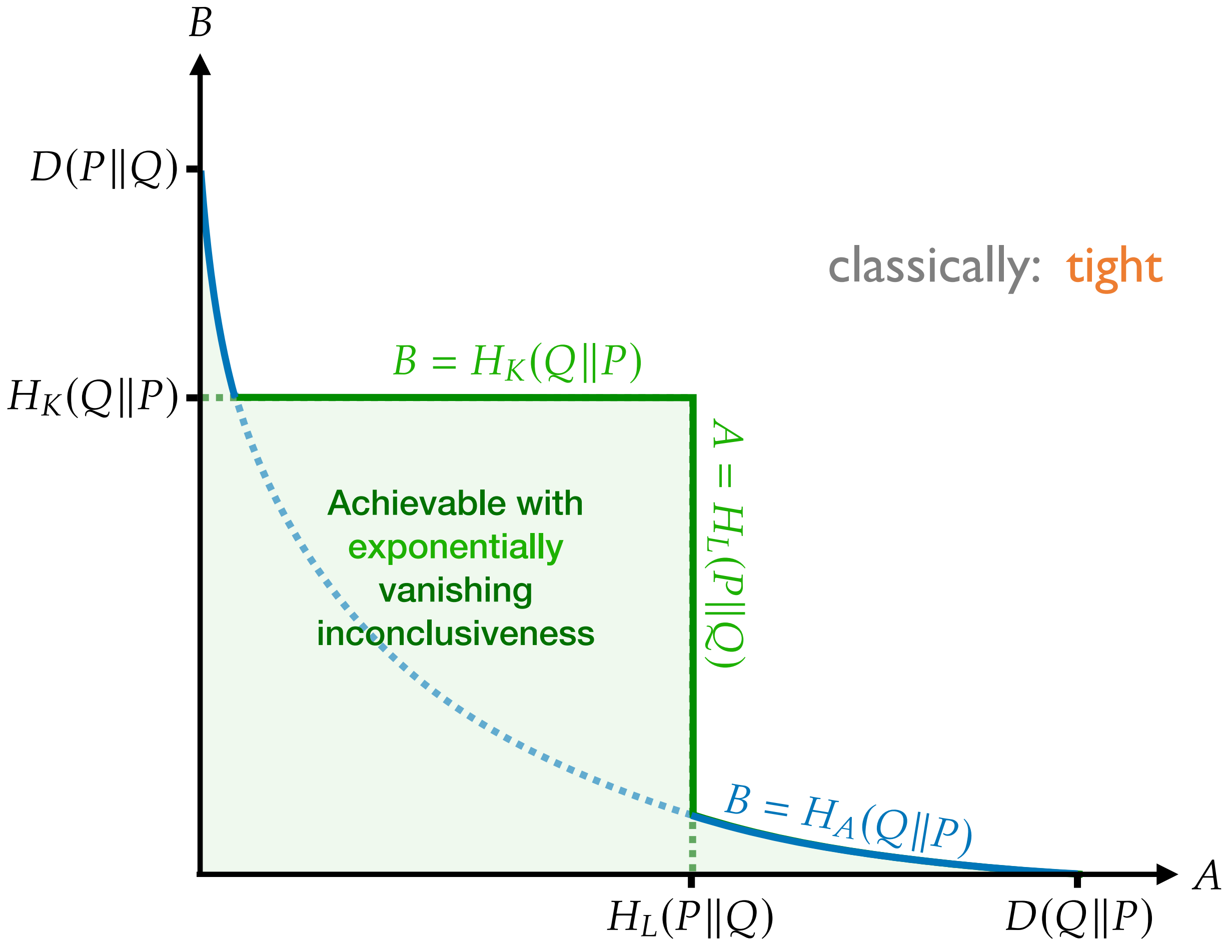
Full range of trade-offs



Exponentially good tests

$$\overline{\alpha}_n \sim \exp(-nA) \qquad \overline{\beta}_n \sim \exp(-nB)$$

$$\pi_n(\rho) \sim 1 - \exp(-nK) \qquad \pi_n(\sigma) \sim 1 - \exp(-nL)$$



classically: tight

converse

$$A \leq H_L(\rho\|\sigma)$$
$$B \leq H_K(\sigma\|\rho)$$

$$H_A(\sigma\|\rho) = \sup_{s\in(0,1)} \frac{s-1}{s}\Big(A - D_s(\sigma\|\rho)\Big)$$

Hoeffding divergence

Exponentially good tests

$$\overline{\alpha}_n \sim \exp(-nA)$$

$$\overline{\beta}_n \sim \exp(-nB)$$

$$\pi_n(\rho) \sim 1 - \exp(-nK)$$

$$\pi_n(\sigma) \sim 1 - \exp(-nL)$$

achievability

$$A \leq \overleftrightarrow{H}_L(\rho\|\sigma)$$

$$B \leq \widetilde{H}_K(\sigma\|\rho)$$

converse

$$A \leq H_L(\rho\|\sigma)$$

$$B \leq H_K(\sigma\|\rho)$$

$$\widetilde{D}_s(\sigma\|\rho) = \frac{1}{s-1} \log \mathrm{Tr}\left(\rho^{\frac{1-s}{2s}} \sigma \rho^{\frac{1-s}{2s}}\right)^s$$

sandwiched Rényi divergence

$$\overleftrightarrow{D}_s(\sigma\|\rho) = \frac{s}{1-s} \widetilde{D}_{1-s}(\rho\|\sigma)$$

reverse sandwiched Rényi divergence

quantumly: converse not tight!

Petz–Rényi not enough???

Symmetric hypothesis testing

$$\mathrm{err}_n = p\,\alpha_n + q\,\beta_n \qquad \mathrm{err}_n \sim \exp(-n\xi(\rho\|\sigma))$$

$$\xi(\rho\|\sigma) = \sup_{s\in(0,1)} (1-s)D_s(\rho\|\sigma)$$

Chernoff bound

$$\overline{\mathrm{err}}_n = p\,\overline{\alpha}_n + q\,\overline{\beta}_n$$

$$\overline{\mathrm{err}}_n \sim \exp(-n\min\{D(\rho\|\sigma),\, D(\sigma\|\rho)\})$$

$$\pi_n(\rho),\, \pi_n(\sigma) \to 1$$

$$\overline{\mathrm{err}}_n = \frac{p\alpha_n + q\beta_n}{\pi_n(p\rho + q\sigma)}$$

$$\overline{\mathrm{err}}_n \sim \exp(-np\max\{D(\rho\|\sigma),\, D(\sigma\|\rho)\})$$

$$\pi_n(p\rho + q\sigma) \to 1$$

Summary

- Conventional Hoeffding and Chernoff bounds can be exceeded with **just a touch of inconclusiveness** — minimal overhead, huge gains
- Further advantages possible by sacrificing conclusiveness: **complete characterisation of trade-offs** achievable and "strong converse" behaviour
- The advantage is robust: even **exponentially good tests** can overcome the limits of Hoeffding and Chernoff

Hoeffding and Chernoff bounds not as fundamental as they seem?

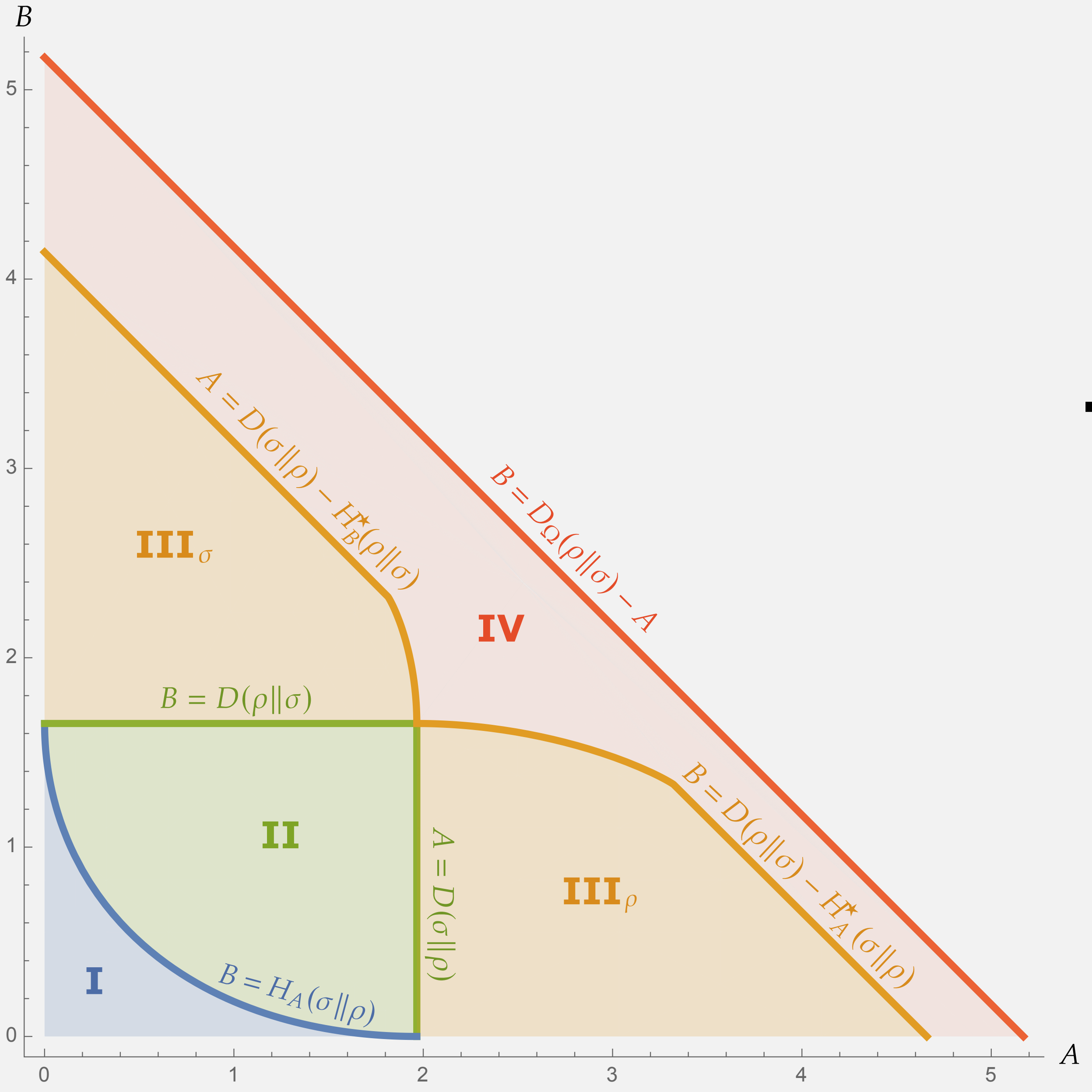
Summary

- Conventional Hoeffding and Chernoff bounds can be exceeded with **just a touch of inconclusiveness** — minimal overhead, huge gains
- Further advantages possible by sacrificing conclusiveness: **complete characterisation of trade-offs** achievable and "strong converse" behaviour
- The advantage is robust: even **exponentially good tests** can overcome the limits of Hoeffding and Chernoff

Hoeffding and Chernoff bounds are **artifacts of demanding perfect conclusiveness**

Open problems

- Exponentially good tests: which Rényi entropy gives optimal behaviour? (Why *not* Petz?)
- Potential extensions: quantum channel discrimination, composite hypothesis testing, …
- Missing technical tools: large deviations for sequential hypothesis testing (non-iid — martingales) 😰



Thank you

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