Emulation Capacity between Idempotent Channels

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If we pay an *even lower* price, can useful information still be transmitted?

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- Emulation Capacity $C(\mathcal{G} \mapsto \mathcal{F})$: best possible rate k/n.
- If $\mathcal{F}$ is the identity : channel coding. [Shannon 1948]
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- For Φ an arbitrary channel, a subsequence of the sequential composition $\Phi^n = \Phi \circ \cdots \circ \Phi$ converges to an idempotent channel
- Capacities of sequential compositions studied in [Fawzi/Rahaman/Taheri 2024], [Singh/Rahaman/Datta 2024], [Singh/Datta 2025]

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- Discrimination of idempotent channels also recently studied [Singh/Bergh 2026]

Structural properties of idempotent channel

If $\mathcal{F} : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H})$ idempotent

$$\mathrm{Rg}(\mathcal{F}) = \left(\bigoplus_{k=1}^{K} \mathcal{L}(\mathcal{H}_{k,1}) \otimes \rho_k \right) \oplus 0$$

- Shape vector of $\mathcal{F}$: $\lambda(\mathcal{F}) = (\dim(\mathcal{H}_{1,1}), \ldots, \dim(\mathcal{H}_{K,1}))$
- $\lambda(\mathcal{F})$ computable in polynomial time [Fawzi/Rahaman/Taheri 2024]
- Stable under tensor product : $\lambda(\mathcal{F} \otimes \mathcal{G}) = \lambda(\mathcal{F}) \otimes \lambda(\mathcal{G})$
- **Idea:** If $\mathcal{F}(x) = x$ for x full-rank, $\mathcal{F}^*$ is a conditional expectation and its range has a known structure [Wolf/Pérez-García 2010]

Emulation capacity - Idempotent channels

Theorem 1 (Emulation Capacity)

For $\mathcal{F}$, $\mathcal{G}$ idempotent:

$$C(\mathcal{G} \mapsto \mathcal{F}) = \inf_{p \in [1,+\infty]} \frac{\log(\|\lambda(\mathcal{G})\|_p)}{\log(\|\lambda(\mathcal{F})\|_p)}.$$

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- Additivity for the source channel: $C(\mathcal{G}_1 \otimes \mathcal{G}_2 \mapsto \mathcal{F}) = C(\mathcal{G}_1 \mapsto \mathcal{F}) + C(\mathcal{G}_2 \mapsto \mathcal{F})$
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Emulation capacity fully determined by the shape vector

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Let $(k_\nu)_{\nu\in\mathbb{N}}$, $(n_\nu)_{\nu\in\mathbb{N}}$ *be two integer sequences with* $\lim_{\nu\to\infty} n_\nu = +\infty$ *such that there exists* $\varepsilon > 0$ *satisfying*

$$\lim_{\nu\to\infty} \frac{k_\nu}{n_\nu} \geq \min_{p\in\{1,+\infty\}} \frac{\log(\|\lambda(\mathcal{G})\|_p)}{\log(\|\lambda(\mathcal{F})\|_p)} + \varepsilon.$$

Then for all sequences $(\mathcal{E}_\nu)_{\nu\in\mathbb{N}}$, $(\mathcal{D}_\nu)_{\nu\in\mathbb{N}}$, $\frac{1}{2}\lim_{\nu\to\infty} \|\mathcal{F}^{\otimes k_\nu} - \mathcal{D}_\nu\mathcal{G}^{\otimes n_\nu}\mathcal{E}_\nu\|_\diamond = 1.$

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- $\|\mathcal{F}\|_\diamond = \sup_{n\in\mathbb{N}^*} \|\mathcal{F}\otimes \mathrm{Id}_n\|_{1\to 1}$
- $\frac{1}{2}\|\Phi - \Psi\|_\diamond = 1$ iff Φ and Ψ maximally distinguishable

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- Known for self-adjoint idempotent channels [Gao/Junge/LaRacuente 2018]

Much better behaved than the general case.

Cases without minimisation

$\mathcal{F}$	$\mathcal{G}$	$C(\mathcal{G} \mapsto \mathcal{F})$
Id_d	$\mathcal{G}$	$\frac{\log(\|\lambda(\mathcal{G})\|_\infty)}{\log(d)}$
$\mathcal{F}$	Id_d	$\frac{\log(d)}{\log(\|\lambda(\mathcal{F})\|_1)}$
Δ_d	$\mathcal{G}$	$\frac{\log(\|\lambda(\mathcal{G})\|_1)}{\log(d)}$
$\mathcal{F}$ s.t. $\lambda(\mathcal{F}) \neq (1, \ldots, 1)$	Δ_d	0
$\mathcal{F}$ s.t. $\lambda(\mathcal{F}) = (1, \ldots, 1)$	Δ_d	$\frac{\log(d)}{\log(\|\lambda(\mathcal{F})\|_1)}$

Table 1: Emulation capacities when either $\mathcal{F}$ or $\mathcal{G}$ is the identity or the completely dephasing channel.

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Table 1: Emulation capacities when either $\mathcal{F}$ or $\mathcal{G}$ is the identity or the completely dephasing channel.

In these cases, the strong-converse rate and the emulation capacity coincide

Sketch of the proof of Theorem 1

- Achievability
  - Amounts to embedding $\mathrm{Rg}(\widehat{\mathcal{F}}^*)$ into $\mathrm{Rg}(\widehat{\mathcal{G}}^*)$ as $*$-algebras
  - Asymptotic embedding of $*$-algebras studied in [Kuperberg 2002]
  - For all $p \in [1, +\infty]$, $\|\lambda(\mathcal{F})\|_p < \|\lambda(\mathcal{G})\|_p \implies$ there is $k, n, \mathcal{D}, \mathcal{E}$ with $\mathcal{F}^{\otimes k} = \mathcal{D}\mathcal{G}^{\otimes n}\mathcal{E}$.
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  - If $\mathcal{F} = \mathcal{D}\mathcal{G}\mathcal{E}$, we can embed $\mathrm{Rg}(\mathcal{F}^*)$ into $\mathrm{Rg}(\mathcal{G}^*)$
    - Embedding given by $x \mapsto e_{\mathcal{G}\mathcal{E}}\mathcal{G}^*\mathcal{D}^*(x)e_{\mathcal{G}\mathcal{E}} = \widetilde{\mathcal{G}}^*\mathcal{D}^*(x)$, with $e_{\mathcal{G}\mathcal{E}} = \mathcal{G}\mathcal{E}(\mathbf{1})^0$.
    - Amounts to show that $\mathrm{Rg}(\mathcal{F}^*) \subseteq M_{\widetilde{\mathcal{G}}^*\mathcal{D}^*}$, multiplicative domain of $\widetilde{\mathcal{G}}^*\mathcal{D}^*$.
    - For Φ CP, $M_{\Phi^*} = \{X \mid \forall Y, \Phi(XY) = \Phi(X)\Phi(Y),\ \Phi(YX) = \Phi(Y)\Phi(X)\}$
    - For Φ Unital CP, i.e Φ^* is CPTP, $M_\Phi = \{X \mid \Phi(X^*X) = \Phi(X^*)\Phi^*(X),\ \Phi(XX^*) = \Phi(X)\Phi(X^*)\}$.
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  - Conclude by multiplicativity of the shape vector.

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- We show that $\frac{1}{2}\|\mathcal{F}-\mathcal{D}\mathcal{G}\mathcal{E}\|_\diamond \geq 1 - \min_{p\in\{1,+\infty\}} \frac{\|\lambda(\mathcal{G})\|_p}{\|\lambda(\mathcal{F})\|_p}$.

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- Holevo-Helstrom: $\frac{1}{2}\|\Phi_1 - \Phi_2\|_\diamond \geq \mathrm{tr}(\mu\Phi_1 \otimes \mathrm{Id}_R(\sigma)) - \mathrm{tr}(\mu\Phi_2 \otimes \mathrm{Id}_R(\sigma))$ for any $0 \leq \mu \leq \mathbf{1}$, σ a state.
- Goal: Find μ, σ such that:
  - $\mathrm{tr}(\mu\mathcal{F} \otimes \mathrm{Id}_R(\sigma)) = 1$,
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- For $p=1$ ('purely classical'): maximally correlated state for σ, μ projector on $\mathrm{supp}(\sigma)$
- For $p=+\infty$ ('purely quantum'): $\mu=\sigma$ is the maximally entangled state.

Rate if a very small error is allowed

- Proof of the converse bound based on *structure properties* of multiplicative domains.
- Multiplicative domains of UCP are $*$-algebras: if $X, Y \in M_\Phi$: $\{X^*, XY, \mathbf{1}\} \subseteq M_\Phi$.
- *Loosen* these requirements up to an ε: $\varepsilon - C*$ algebras introduced in [Kitaev 2025]
- We show that Φ^* UCP is *almost* multiplicative on $\{X \mid \|\Phi^*(x^*x) - \Phi^*(x^*)\Phi^*(x)\| \leq \delta\|x\|^2, \ \|\Phi^*(xx^*) - \Phi^*(x)\Phi^*(x^*)\| \leq \delta\|x\|^2\}$.
- Conclude that if $\|\mathcal{F}^{\otimes k} - \mathcal{D}\mathcal{G}^{\otimes n}\mathcal{E}\|_\diamond$ is *very* small, $\frac{k}{n} \leq \inf_{p\in[1,+\infty]} \frac{\log(\|\lambda(\mathcal{G})\|_p)}{\log(\|\lambda(\mathcal{F})\|_p)}$.
- Theorem 1 follows for $\|\mathcal{F}^{\otimes k} - \mathcal{D}\mathcal{G}^{\otimes n}\mathcal{E}\|_\diamond = 0$.
- Slight improvement of the single shot case

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Open problem
Is

$$\inf_{p \in [1,+\infty]} \frac{\log(\|\mathcal{F}\|_p)}{\log(\|\mathcal{G}\|_p)}$$

a strong-converse rate for the emulation of any idempotent channels?