An almost complete picture of quantum hypothesis testing with composite correlated hypotheses

arXiv:2508.12901 (IEEE TIT) & 2508.12889

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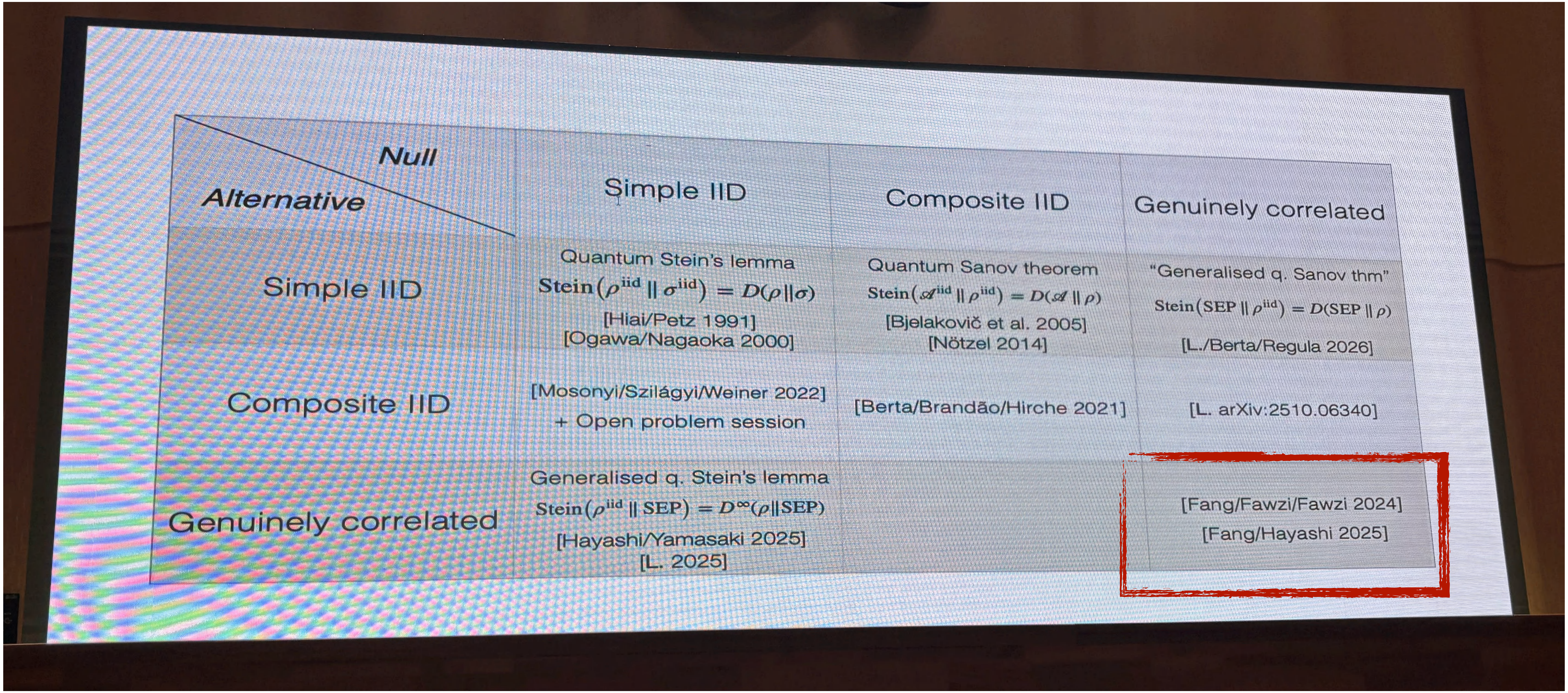
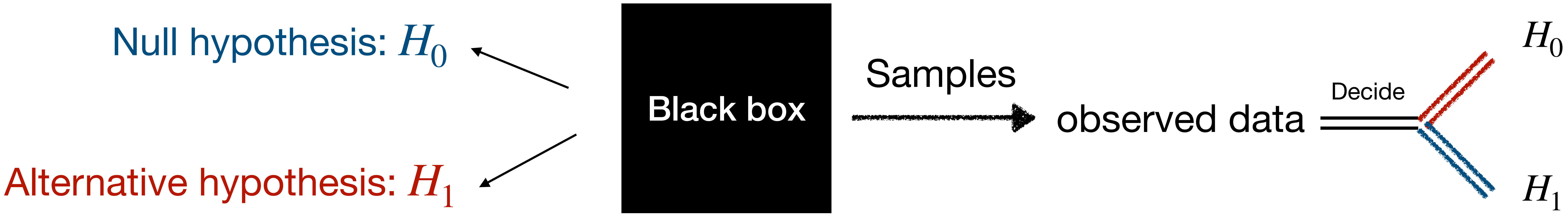
Beyond IID 2026 @ CUHK-Shenzhen



Null / ***Alternative***	Simple IID	Composite IID	Genuinely correlated
Simple IID	Quantum Stein's lemma $\mathrm{Stein}(\rho^{\mathrm{iid}} \,\|\, \sigma^{\mathrm{iid}}) = D(\rho \| \sigma)$ [Hiai/Petz 1991] [Ogawa/Nagaoka 2000]	Quantum Sanov theorem $\mathrm{Stein}(\mathscr{A}^{\mathrm{iid}} \,\|\, \rho^{\mathrm{iid}}) = D(\mathscr{A} \,\|\, \rho)$ [Bjelaković et al. 2005] [Nötzel 2014]	"Generalised q. Sanov thm" $\mathrm{Stein}(\mathrm{SEP} \,\|\, \rho^{\mathrm{iid}}) = D(\mathrm{SEP} \,\|\, \rho)$ [L./Berta/Regula 2026]
Composite IID	[Mosonyi/Szilágyi/Weiner 2022] + Open problem session	[Berta/Brandão/Hirche 2021]	[L. arXiv:2510.06340]
Genuinely correlated	Generalised q. Stein's lemma $\mathrm{Stein}(\rho^{\mathrm{iid}} \,\|\, \mathrm{SEP}) = D^{\infty}(\rho \| \mathrm{SEP})$ [Hayashi/Yamasaki 2025] [L. 2025]		[Fang/Fawzi/Fawzi 2024] [Fang/Hayashi 2025]

What is “hypothesis testing”?



Null hypothesis: H_0

Alternative hypothesis: H_1

Black box

Samples

observed data

Decide

H_0

H_1

A rigorous framework for deciding between competing explanations for observed data.
Cornerstone of statistics, information theory, signal processing, machine learning, and experimental physics…

H_0: Healthy

H_1: Sick

Samples

Decide

Negative

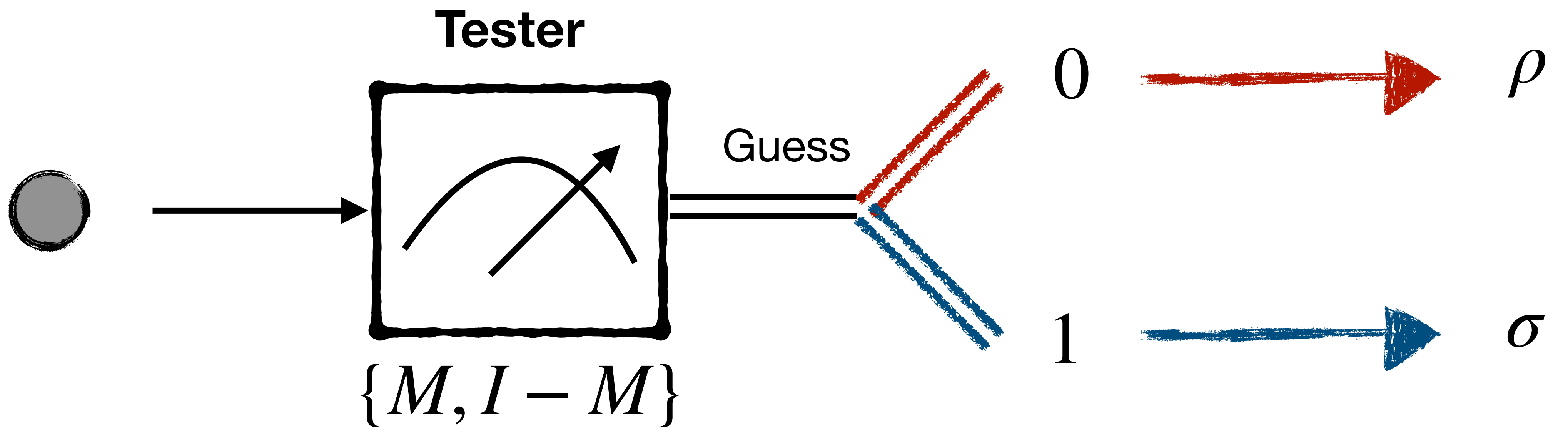
Positive

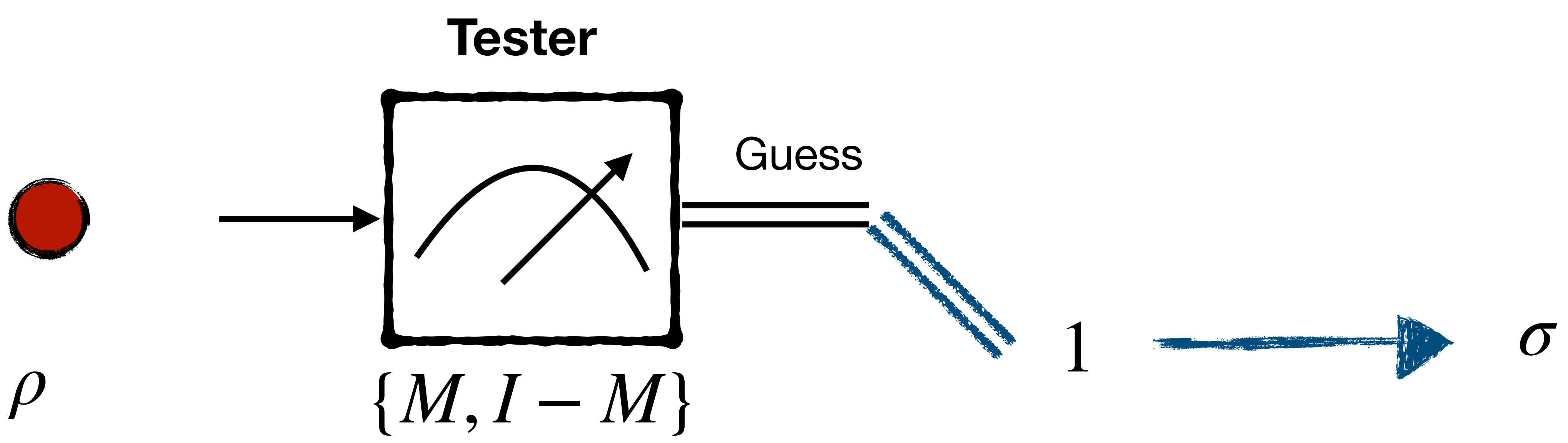
Primary objective: **identify** the best **model**, while **minimizing the probabilities of errors** (i.e., type-I and type-II errors).

What is "quantum hypothesis testing"?

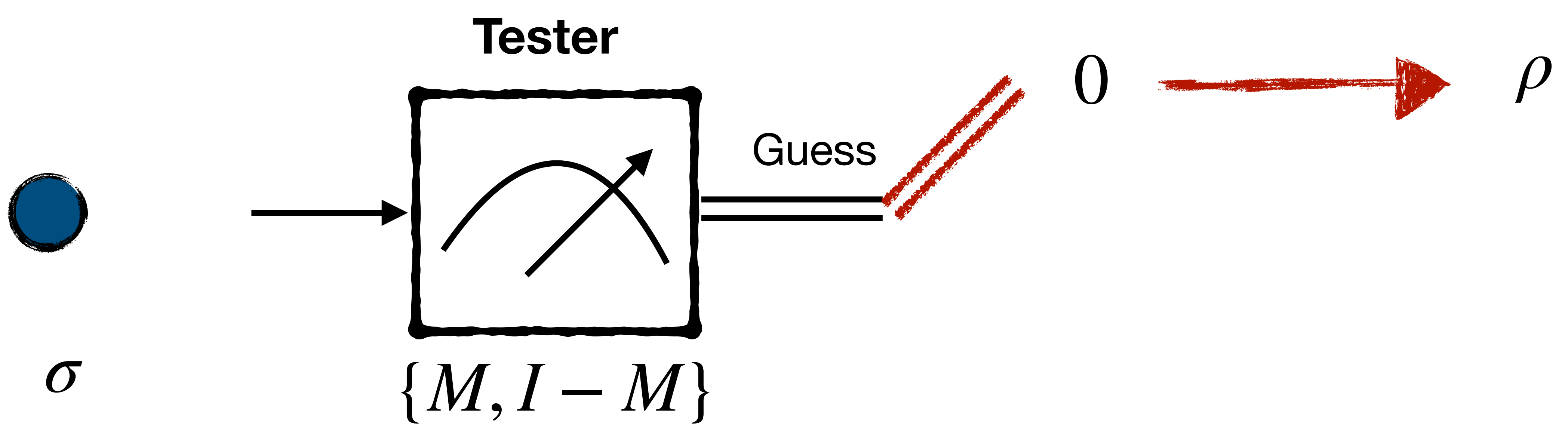
Null hypothesis: ρ

Alternative hypothesis: σ





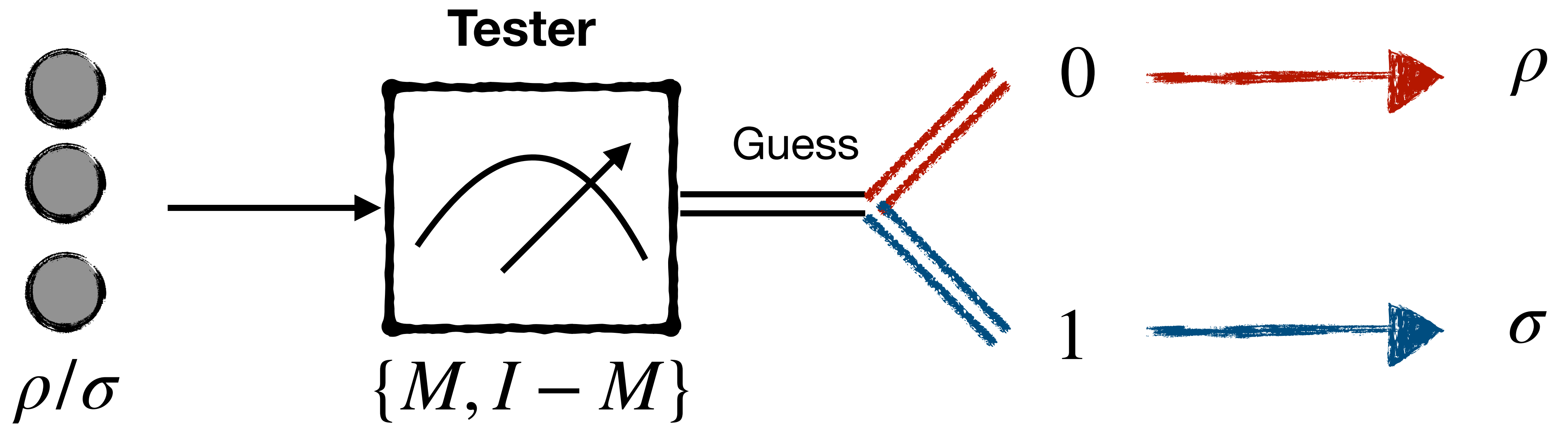
Type-I error $\alpha(\rho, M) := \mathrm{Tr}\,[\rho(I-M)]$

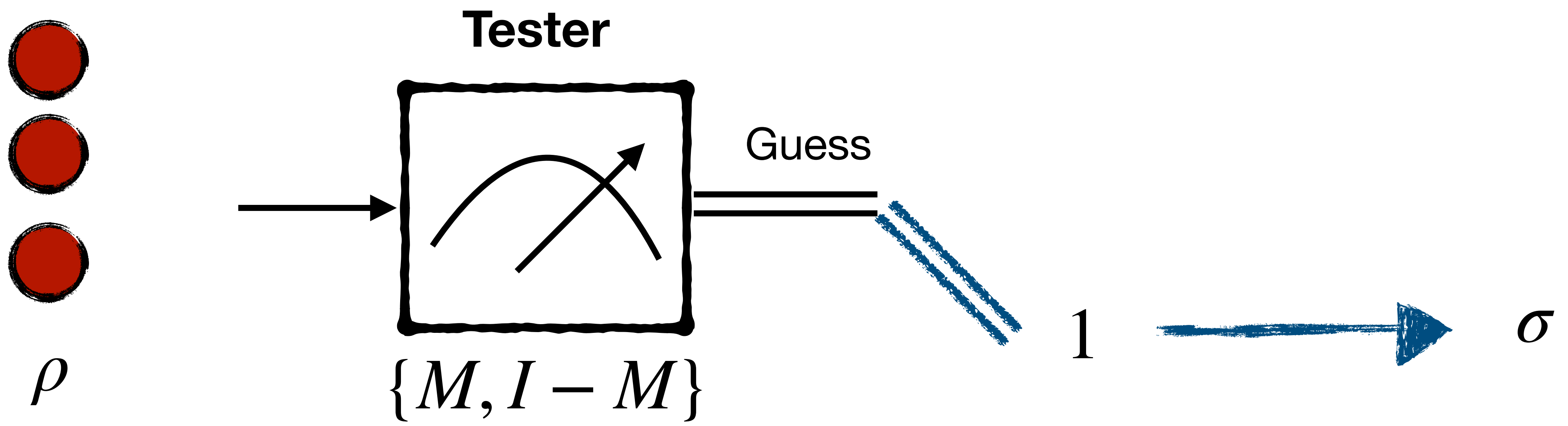


Type-II error $\beta(\sigma, M) := \mathrm{Tr}\,[\sigma M]$

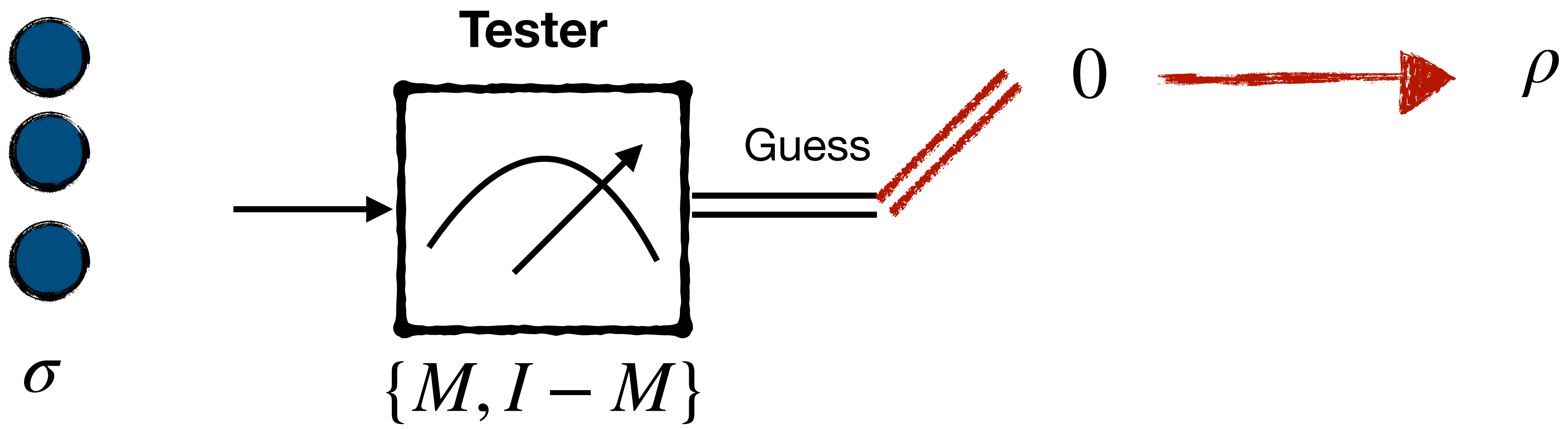
Primary objective: **identify** the best **model**, while **minimizing the probabilities of errors** (i.e., type-I and type-II errors).

What is “quantum hypothesis testing”?





Type-I error $\alpha(\rho^{\otimes n}, M_n) := \mathrm{Tr}\,[\rho^{\otimes n}(I - M_n)]$

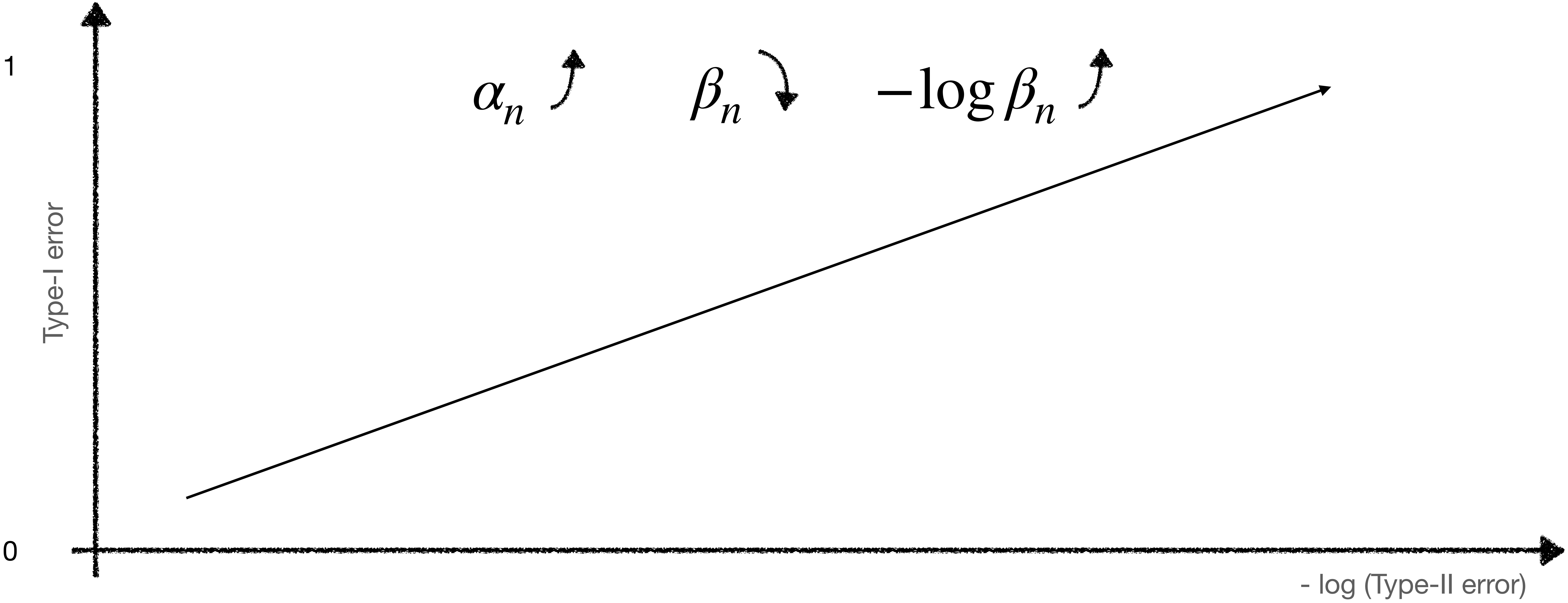


Type-II error $\beta(\sigma^{\otimes n}, M_n) := \mathrm{Tr}\,[\sigma^{\otimes n} M_n]$

Primary objective: **identify** the best **model**, while **minimizing the probabilities of errors** (i.e., type-I and type-II errors).

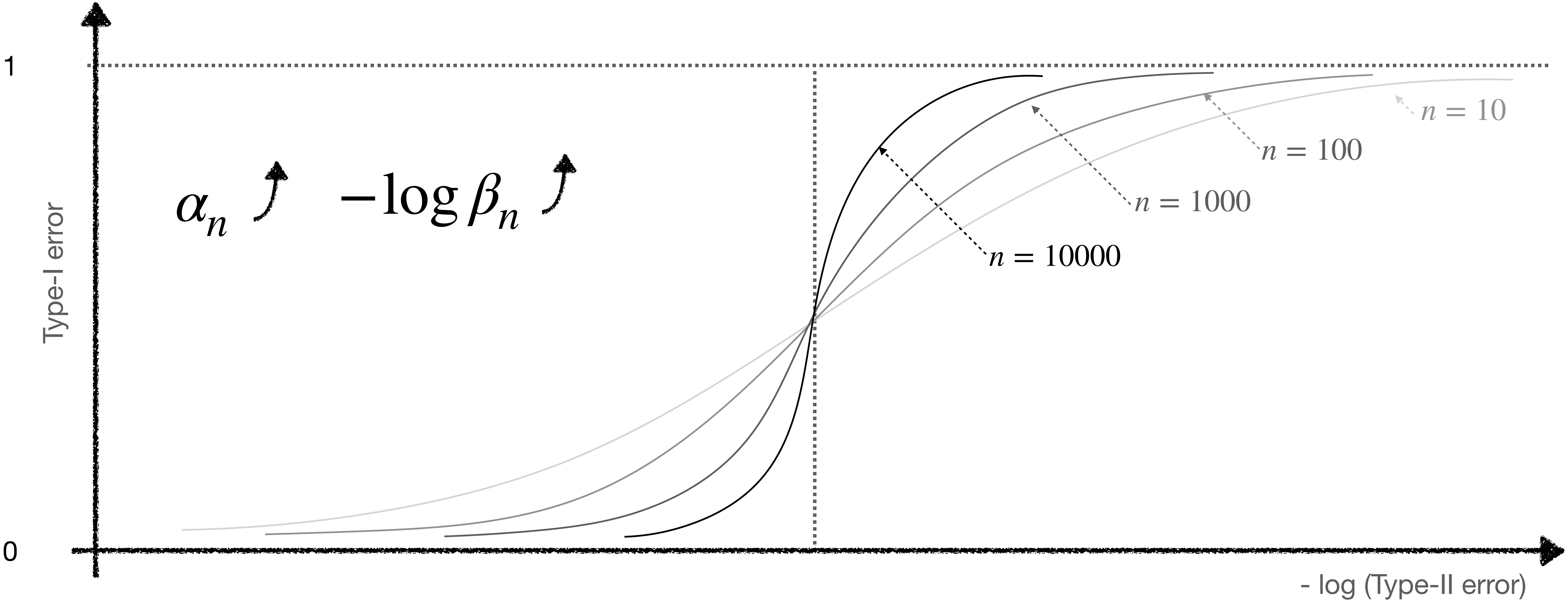
Different operational regimes for error tradeoff

Identify the best **model**, while **minimizing the probabilities of errors**



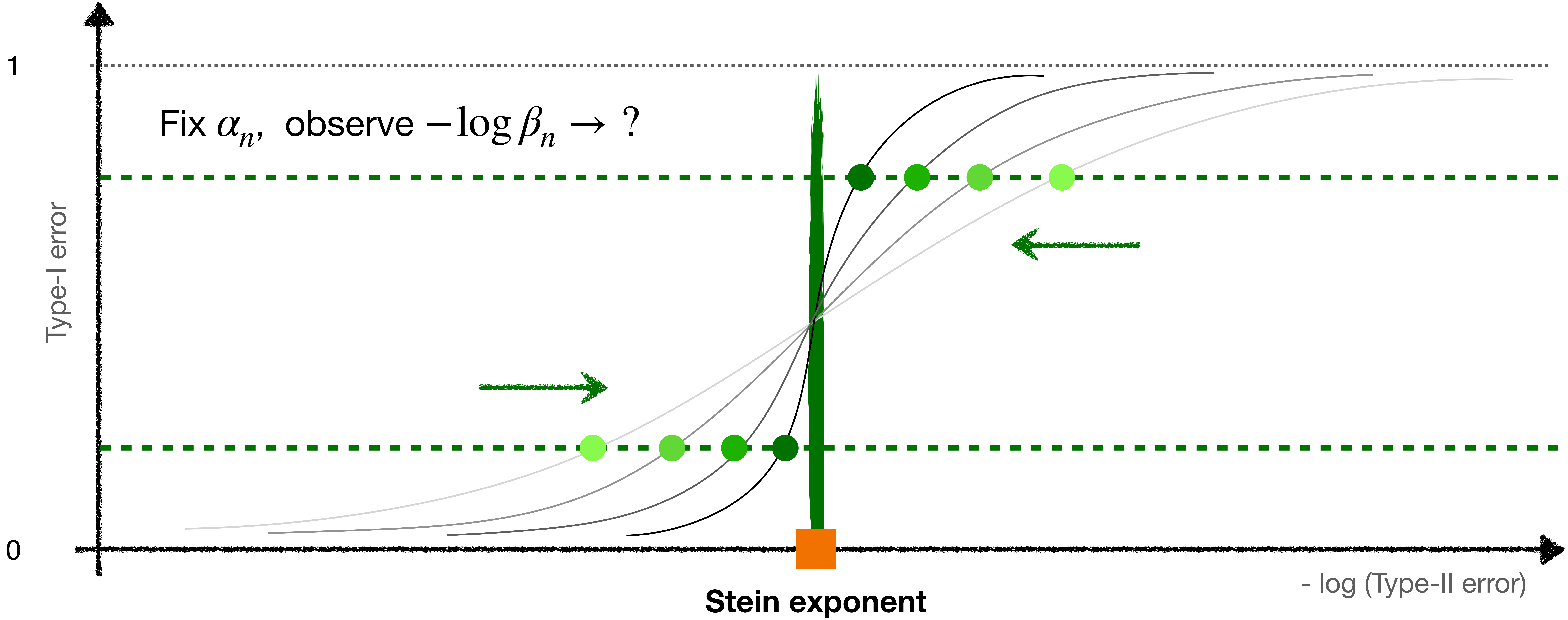
Different operational regimes for error tradeoff

Identify the best **model**, while **minimizing the probabilities of errors**



Different operational regimes for error tradeoff

Identify the best **model**, while **minimizing the probabilities of errors**



Different operational regimes for error tradeoff

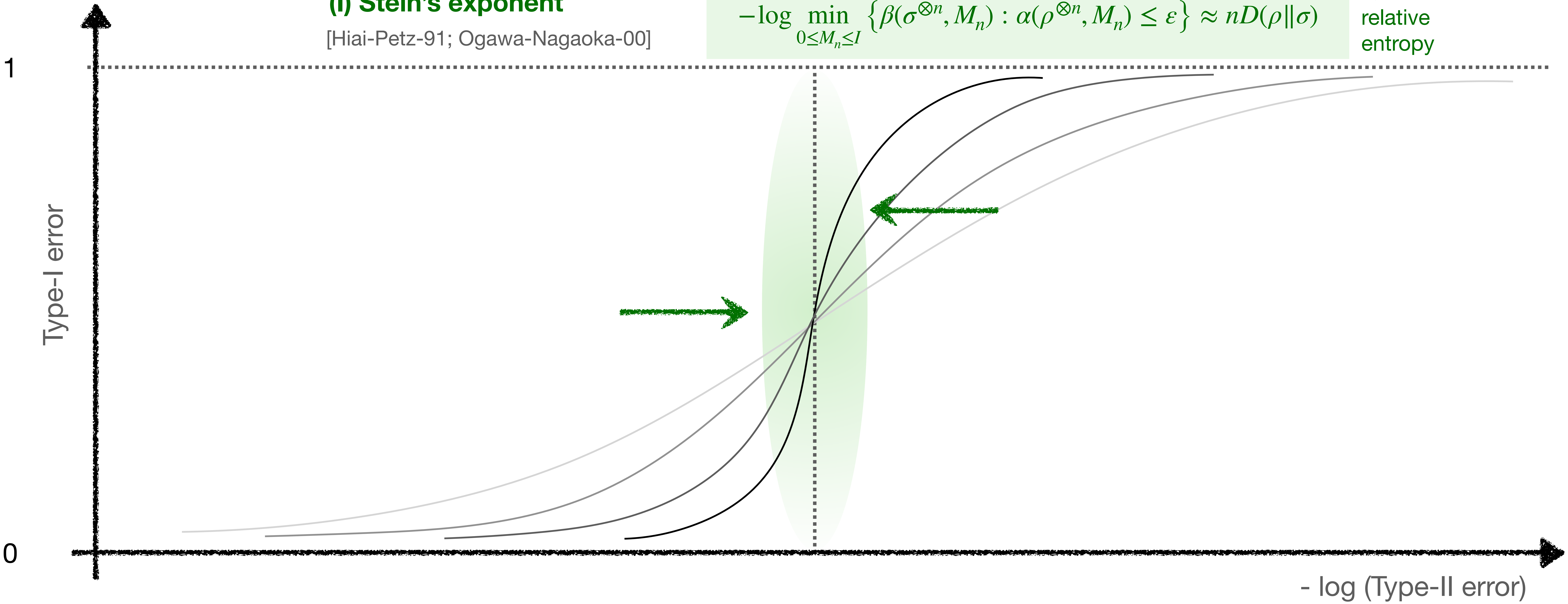
Identify the best **model**, while **minimizing the probabilities of errors**

(I) Stein's exponent

[Hiai-Petz-91; Ogawa-Nagaoka-00]

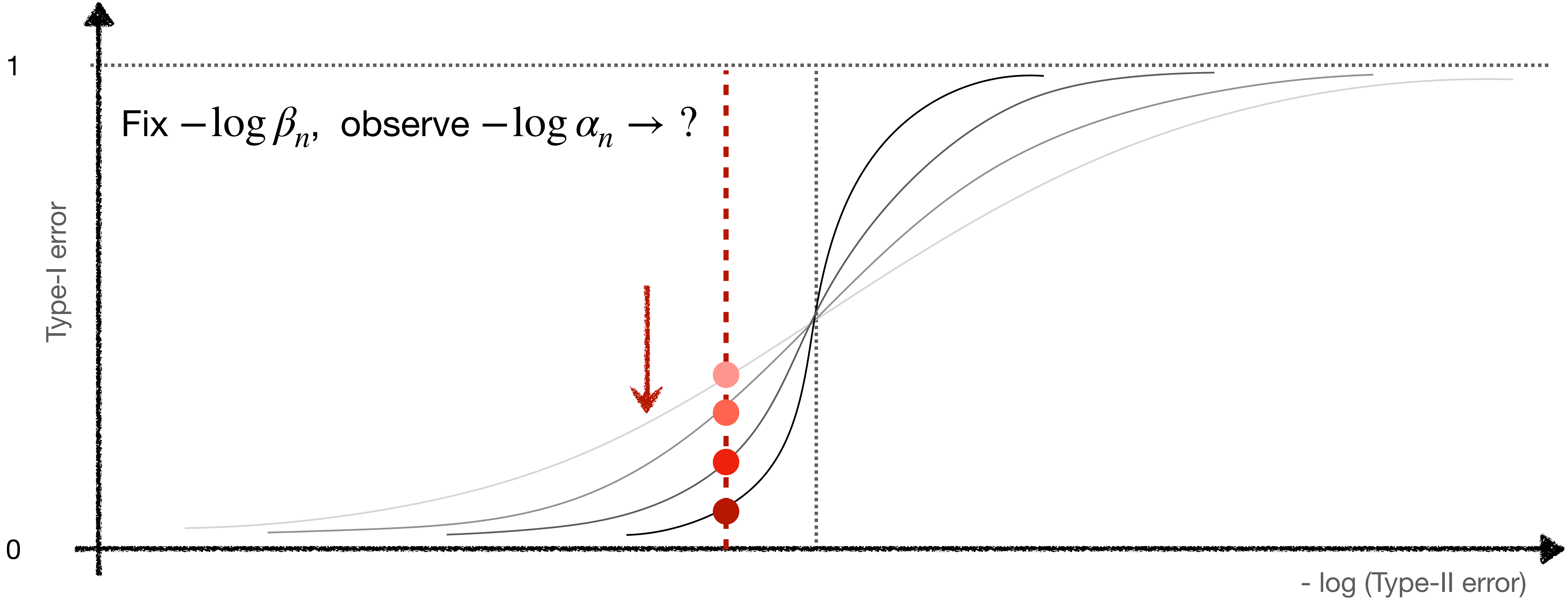
$$-\log \min_{0\leq M_n\leq I} \left\{\beta(\sigma^{\otimes n}, M_n) : \alpha(\rho^{\otimes n}, M_n) \leq \varepsilon\right\} \approx nD(\rho\|\sigma)$$

Quantum
relative
entropy



Different operational regimes for error tradeoff

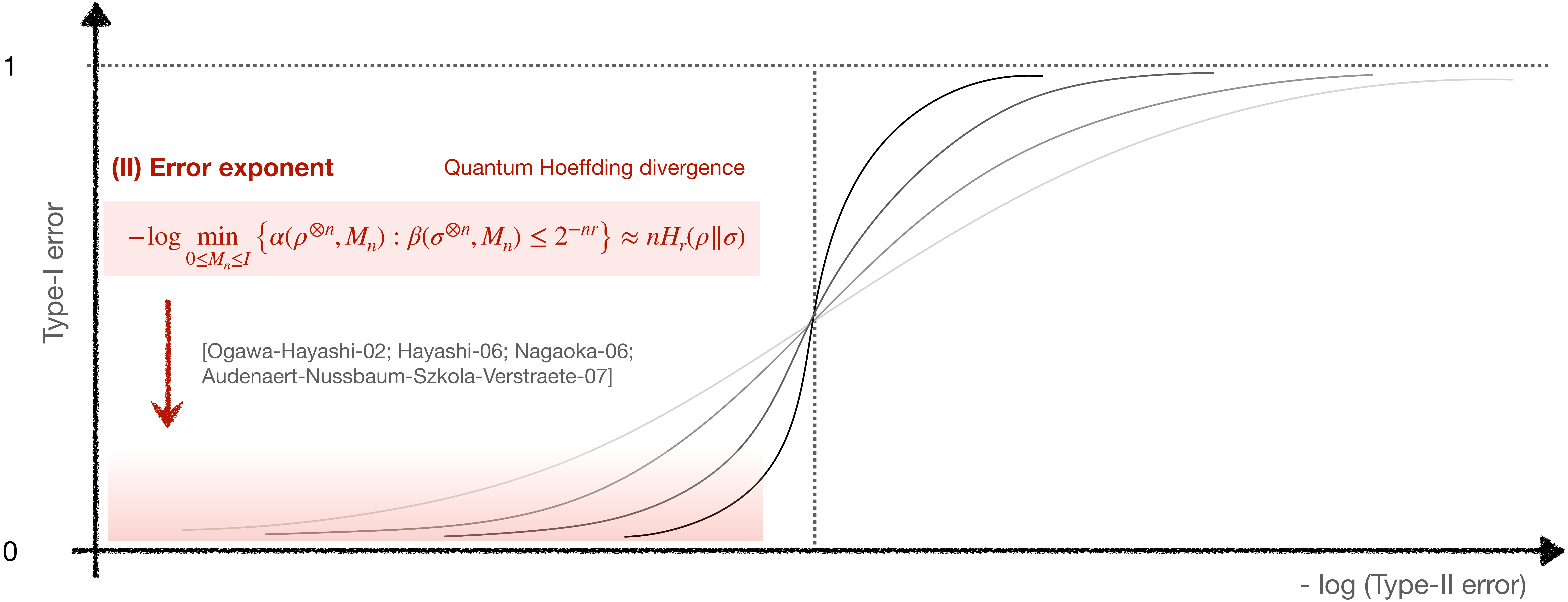
Identify the best **model**, while **minimizing the probabilities of errors**



Fix $-\log \beta_n$, observe $-\log \alpha_n \to ?$

Different operational regimes for error tradeoff

Identify the best **model**, while **minimizing the probabilities of errors**



(II) Error exponent

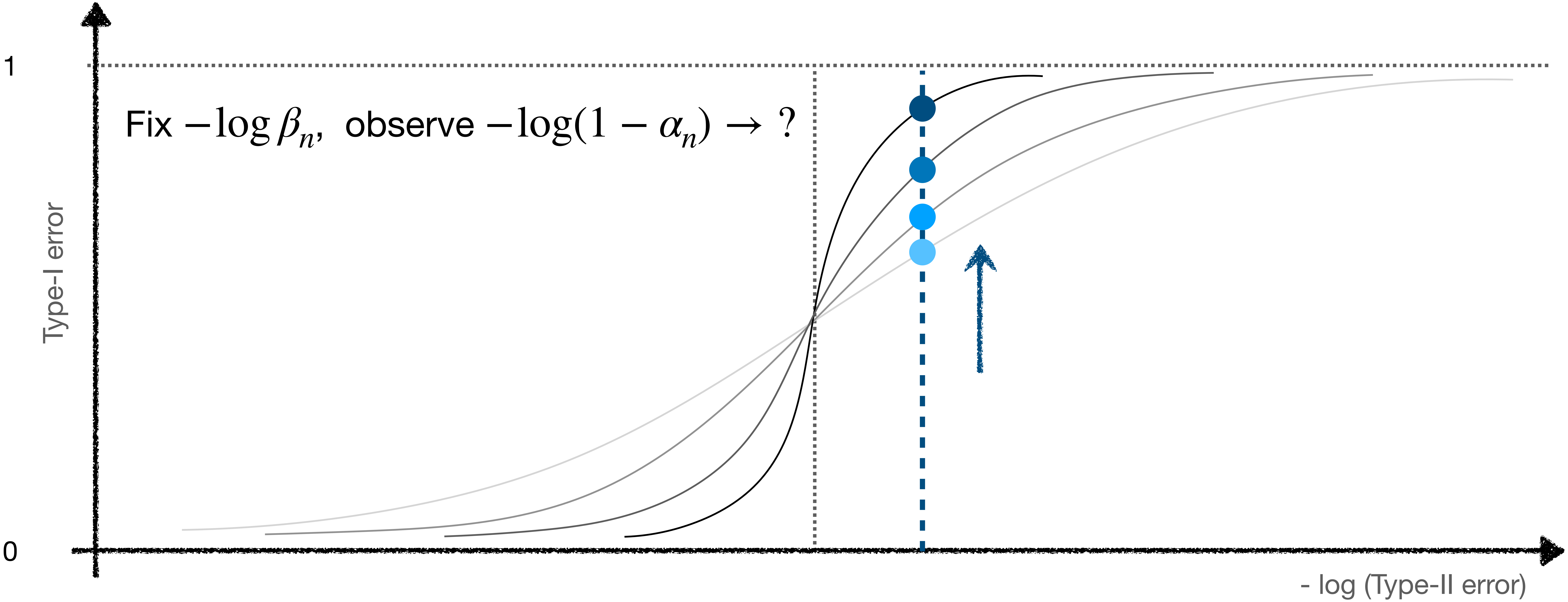
Quantum Hoeffding divergence

$$-\log \min_{0\le M_n \le I} \left\{ \alpha(\rho^{\otimes n}, M_n) : \beta(\sigma^{\otimes n}, M_n) \le 2^{-nr} \right\} \approx nH_r(\rho\|\sigma)$$

[Ogawa-Hayashi-02; Hayashi-06; Nagaoka-06; Audenaert-Nussbaum-Szkola-Verstraete-07]

Different operational regimes for error tradeoff

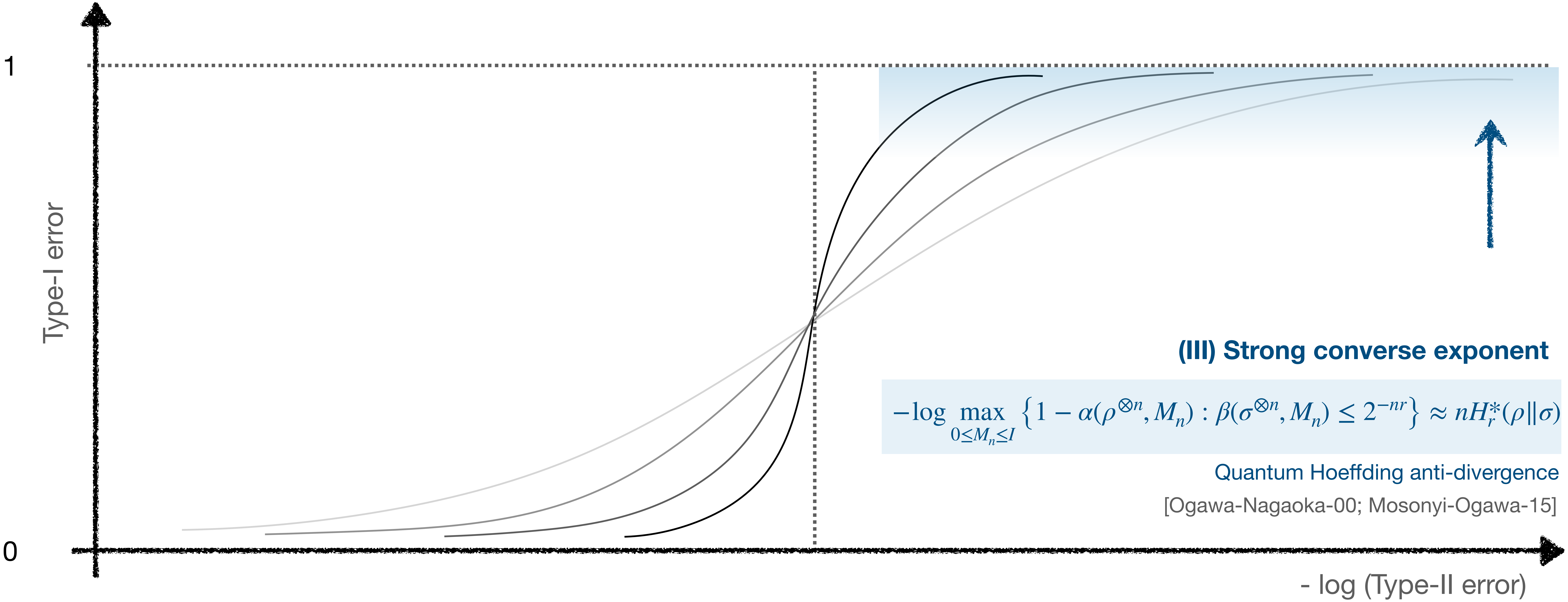
Identify the best **model**, while **minimizing the probabilities of errors**



Fix $-\log \beta_n$, observe $-\log(1-\alpha_n) \to ?$

Different operational regimes for error tradeoff

Identify the best **model**, while **minimizing the probabilities of errors**



(III) Strong converse exponent

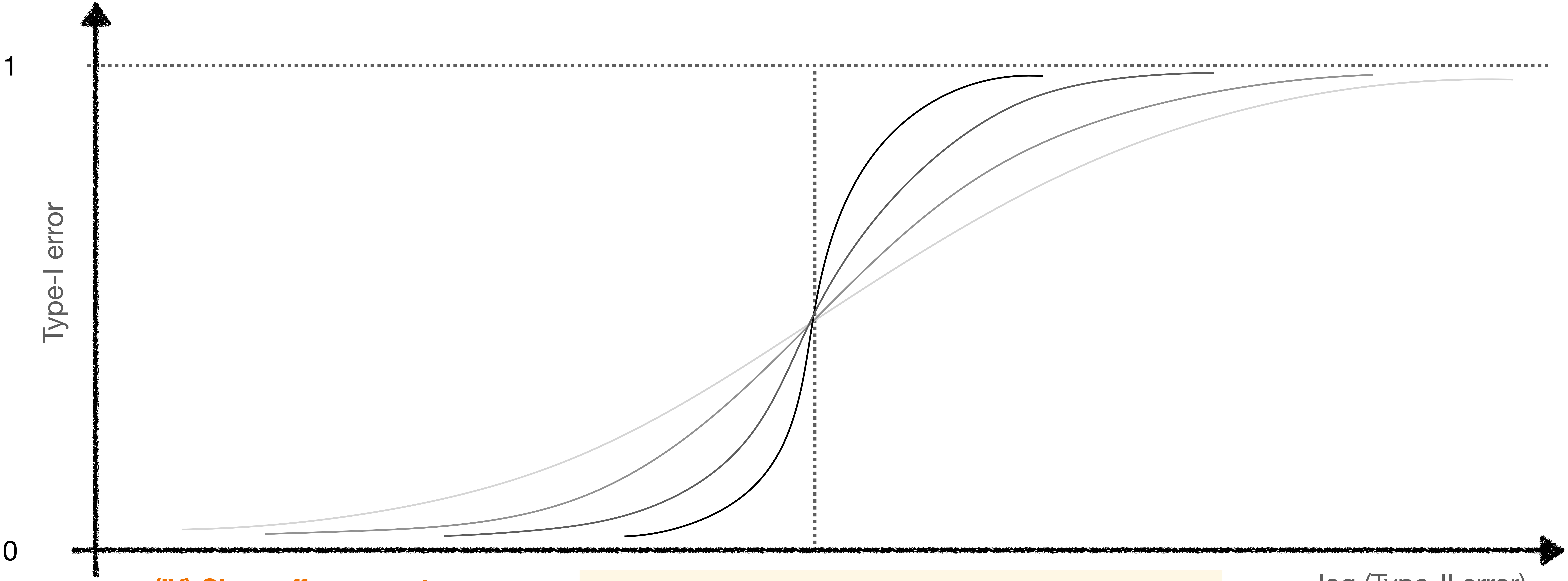
$$-\log \max_{0\leq M_n\leq I} \left\{1-\alpha(\rho^{\otimes n},M_n) : \beta(\sigma^{\otimes n},M_n)\leq 2^{-nr}\right\} \approx nH_r^*(\rho\|\sigma)$$

Quantum Hoeffding anti-divergence

[Ogawa-Nagaoka-00; Mosonyi-Ogawa-15]

Different operational regimes for error tradeoff

Identify the best **model**, while **minimizing the probabilities of errors**



(IV) Chernoff exponent

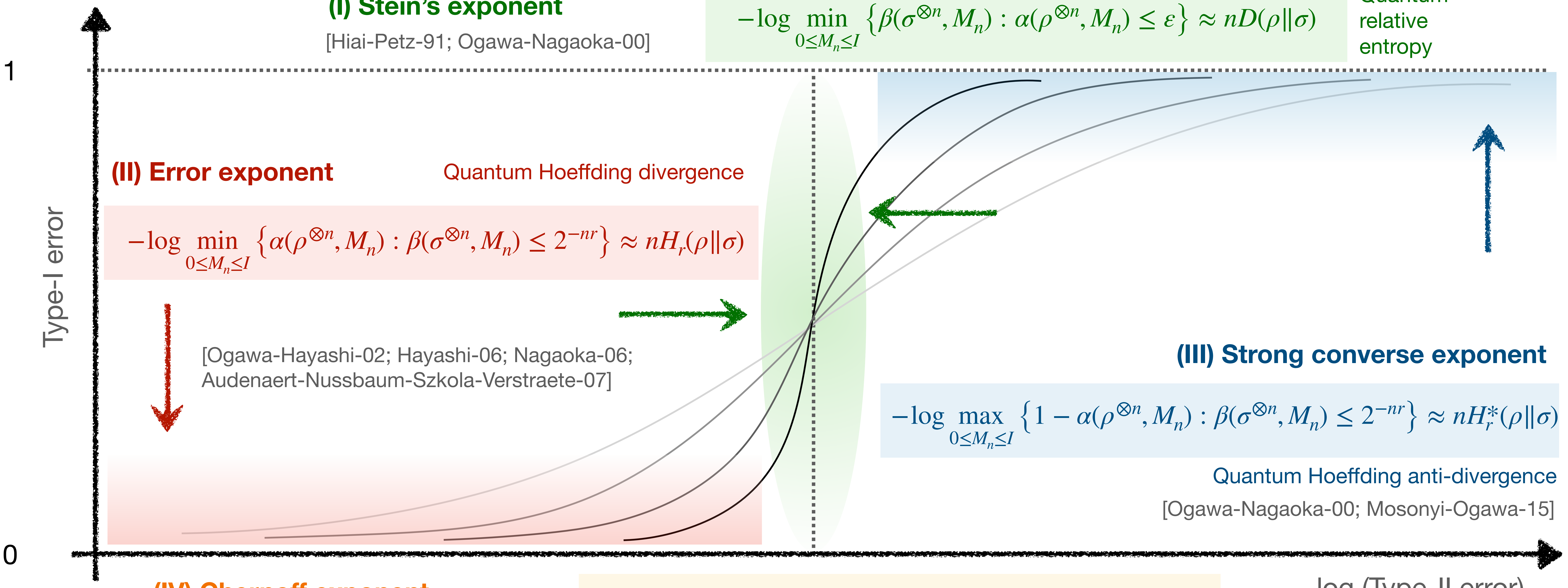
[Audenaert et al.-07; Nussbaum-Szkoła-09]

$$-\log \min_{0\le M_n\le I} \big(\pi_0\alpha(\rho^{\otimes n}, M_n)) + \pi_1\beta(\sigma^{\otimes n}, M_n)\big) \approx nC(\rho\|\sigma)$$

Quantum Chernoff divergence

Different operational regimes for error tradeoff

Identify the best **model**, while **minimizing the probabilities of errors**



(I) Stein's exponent

[Hiai-Petz-91; Ogawa-Nagaoka-00]

$$-\log \min_{0\le M_n\le I} \left\{\beta(\sigma^{\otimes n}, M_n) : \alpha(\rho^{\otimes n}, M_n) \le \varepsilon\right\} \approx nD(\rho\|\sigma)$$

Quantum relative entropy

(II) Error exponent

Quantum Hoeffding divergence

$$-\log \min_{0\le M_n\le I} \left\{\alpha(\rho^{\otimes n}, M_n) : \beta(\sigma^{\otimes n}, M_n) \le 2^{-nr}\right\} \approx nH_r(\rho\|\sigma)$$

[Ogawa-Hayashi-02; Hayashi-06; Nagaoka-06; Audenaert-Nussbaum-Szkola-Verstraete-07]

(III) Strong converse exponent

$$-\log \max_{0\le M_n\le I} \left\{1-\alpha(\rho^{\otimes n}, M_n) : \beta(\sigma^{\otimes n}, M_n) \le 2^{-nr}\right\} \approx nH_r^{*}(\rho\|\sigma)$$

Quantum Hoeffding anti-divergence

[Ogawa-Nagaoka-00; Mosonyi-Ogawa-15]

(IV) Chernoff exponent

[Audenaert et al.-07; Nussbaum-Szkoła-09]

$$-\log \min_{0\le M_n\le I} \big(\pi_0\alpha(\rho^{\otimes n}, M_n)) + \pi_1\beta(\sigma^{\otimes n}, M_n)\big) \approx nC(\rho\|\sigma)$$

Quantum Chernoff divergence

Different operational regimes for error tradeoff

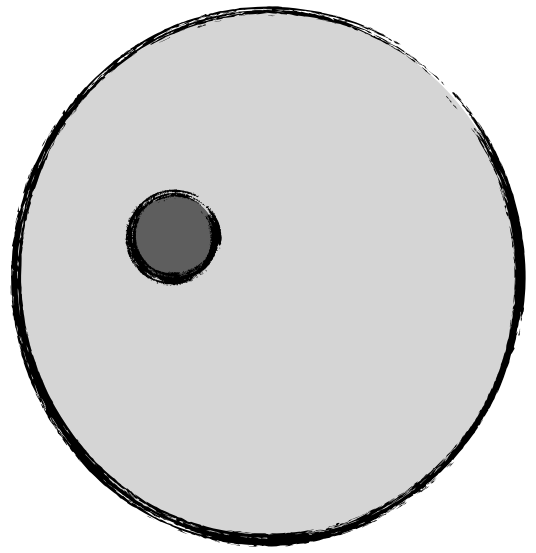
Identify the best **model**, while **minimizing the probabilities of errors**

(I) Stein's exponent	Quantum relative entropy	[Hiai-Petz-91; Ogawa-Nagaoka-00]
(II) Error exponent	Quantum Hoeffding divergence	[Ogawa-Hayashi-02; Hayashi-06; Nagaoka-06; Audenaert-Nussbaum-Szkola-Verstraete-07]
(III) Strong converse exponent	Quantum Hoeffding anti-divergence	[Ogawa-Nagaoka-00; Mosonyi-Ogawa-15]
(IV) Chernoff exponent	Quantum Chernoff divergence	[Audenaert et al.-07; Nussbaum-Szkoła-09]

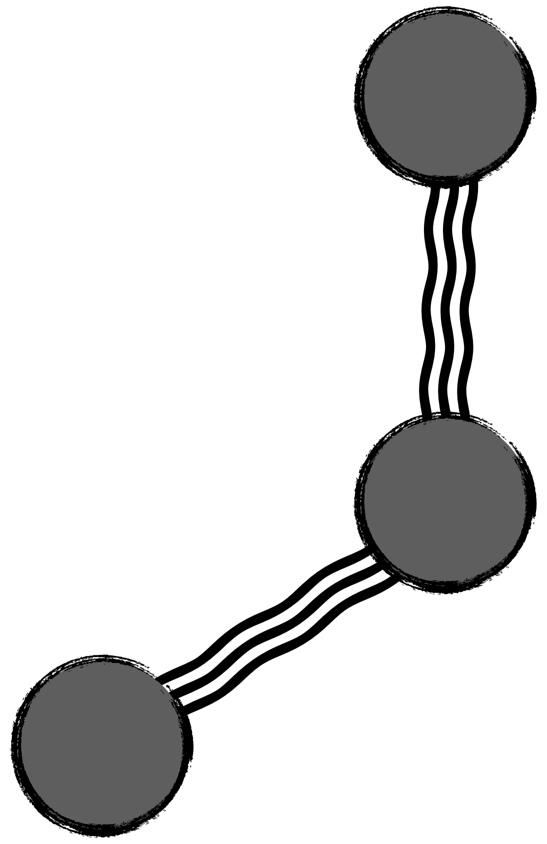
A complete understanding of quantum hypothesis testing for i.i.d. sources developed through a sequence of works across 20+ years

What about “beyond i.i.d.” sources?

While much of the existing literature has focused on i.i.d. sources, practical scenarios often involve



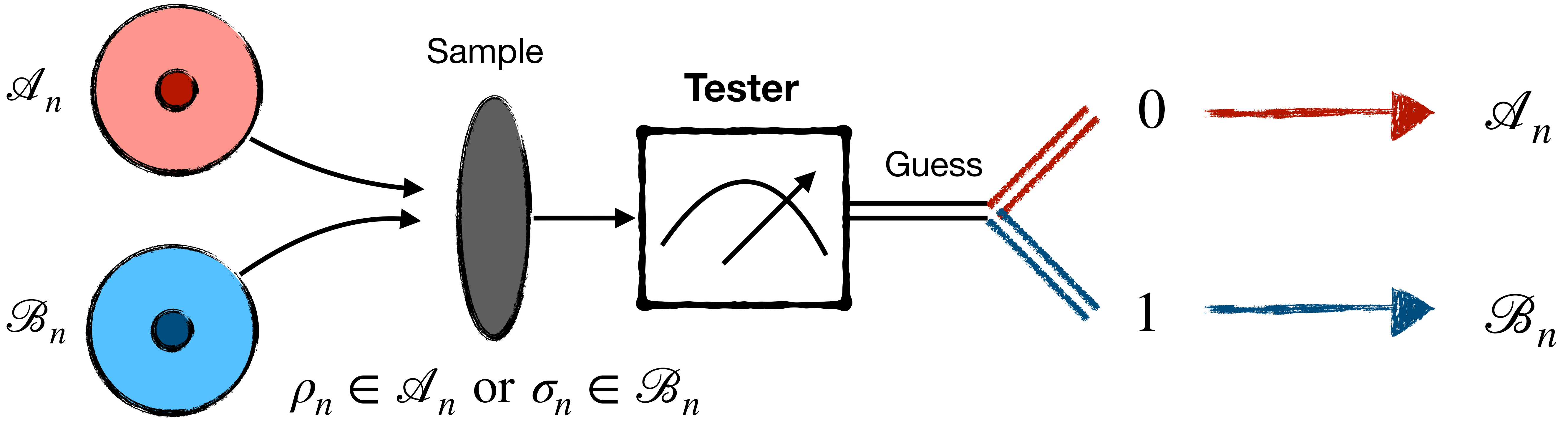
- **Composite hypotheses**: states are not fully specified
  - [Brandao-Plenio-10]: $\rho^{\otimes n}$ v.s. SEP;
  - [Berta-Brandao-Hirche-21]: $\left\{ \int \rho^{\otimes n} d\nu(\rho) \,|\, \rho \in S \right\}$ v.s. $\left\{ \int \sigma^{\otimes n} d\mu(\sigma) \,|\, \sigma \in T \right\}$;
  - [Mosonyi-Szilagyi-Weiner-22]: $\{\rho^{\otimes n} \,|\, \rho \in S\}$ v.s. $\{\sigma^{\otimes n} \,|\, \sigma \in T\}$
  - Many more in Lami’s talk



- **Correlated hypotheses**: states are correlated
  - [Hiai-Mosonyi-Ogawa-07,08]: ρ_n v.s. σ_n, correlated states on a spin chain
  - [Mosonyi-Ogawa-15]: ρ_n v.s. σ_n, correlated states on a spin chain

What is "Composite Correlated" sources?

A tester draws samples from *two sets of correlated quantum states*,
and performs measurements to determine which set the sample belongs to.



(I) Stein's exponent ?

(II) Error exponent?

(III) Strong converse exponent?

(IV) Chernoff exponent?

Type-I error $\alpha(\mathscr{A}_n, M_n) := \sup_{\rho_n \in \mathscr{A}_n} \mathrm{Tr}\,[\rho_n(I - M_n)]$

Worst-case

Type-II error $\beta(\mathscr{B}_n, M_n) := \sup_{\sigma_n \in \mathscr{B}_n} \mathrm{Tr}\,[\sigma_n M_n]$

What is “Composite Correlated” sources?

A tester draws samples from *two sets of correlated quantum states*,
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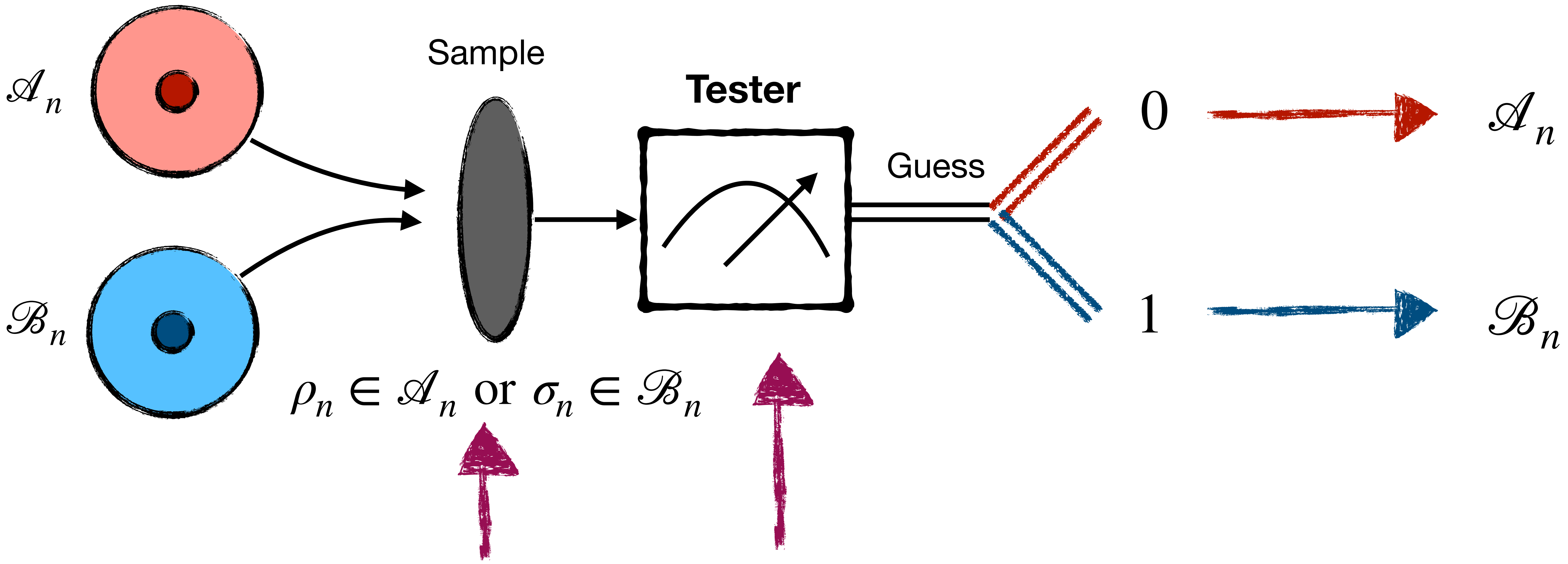


non-i.i.d. structure

state-agnostic test

Minimax opt.: Max over states v.s. Min over test

Challenges

(I) Stein’s exponent ?

(II) Error exponent?

(III) Strong converse exponent?

(IV) Chernoff exponent?

What is "Composite Correlated" sources?

A tester draws samples from *two sets of correlated quantum states*,
and performs measurements to determine which set the sample belongs to.

(I) Stein's exponent (recent progress)

[Hayashi-Yamasaki-24]	$\rho^{\otimes n}$	$\mathscr{B}_n$	A.1		A.3		A.5	
[Lami-24]	$\rho^{\otimes n}$	$\mathscr{B}_n$	A.1	A.2	A.3		A.5	A.6
[KF-Fawzi-Fawzi-24]	$\mathscr{A}_n$	$\mathscr{B}_n$	A.1	A.2	A.3	A.4		

See more progress in Lami's talk

(A.1) Each $\mathscr{B}_n$ is convex and compact;

(A.2) Each $\mathscr{B}_n$ is permutation-invariant;

(A.3) $\mathscr{B}_m \otimes \mathscr{B}_k \subseteq \mathscr{B}_{m+k}$, for all $m, k \in \mathbb{N}$;

(A.5) $\mathscr{B}_1$ contains a full-rank state

(A.6) Each $\mathscr{B}_n$ is closed under partial traces

(A.4) $(\mathscr{B}_m)^\circ_+ \otimes (\mathscr{B}_k)^\circ_+ \subseteq (\mathscr{B}_{m+k})^\circ_+$, for all $m, k \in \mathbb{N}$;

Extending the Stein exponent to more general setting and developing **new applications** across

- quantum cryptography [KF-Fawzi-Fawzi-24]
- quantum communication [Cao-Yao-Berta-25]
- quantum resource theory [KF-Fawzi-Fawzi-24]

What is “Composite Correlated” sources?

A tester draws samples from *two sets of correlated quantum states*,
and performs measurements to determine which set the sample belongs to.

(I) Stein’s exponent (recent progress)

This talk

(II) Error exponent (2508.12901)

(III) Strong converse exponent (2508.12901)

(IV) Chernoff exponent (2508.12889)

an almost complete picture

(A.1) Each $\mathscr{B}_n$ is convex and compact;

~~(A.2) Each $\mathscr{B}_n$ is permutation-invariant;~~

(A.3) $\mathscr{B}_m \otimes \mathscr{B}_k \subseteq \mathscr{B}_{m+k}$, for all $m, k \in \mathbb{N}$;

~~(A.5) $\mathscr{B}_1$ contains a full rank state~~

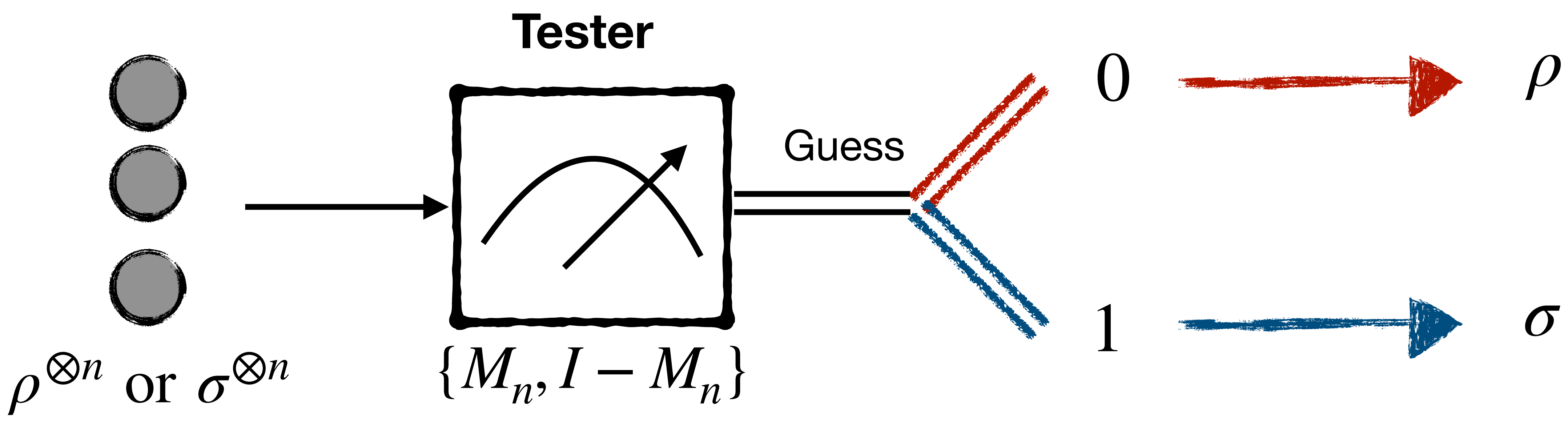
~~(A.6) Each $\mathscr{B}_n$ is closed under partial traces~~

~~(A.4) $(\mathscr{B}_m)_+^\circ \otimes (\mathscr{B}_k)_+^\circ \subseteq (\mathscr{B}_{m+k})_+^\circ$, for all $m, k \in \mathbb{N}$;~~

Only two assumptions (A.1) and (A.3) are required !!!

Unexpected: in the iid case, the proofs for (II), (III), (IV) are much more complicated than (I).

(II) Error exponent (2508.12901)



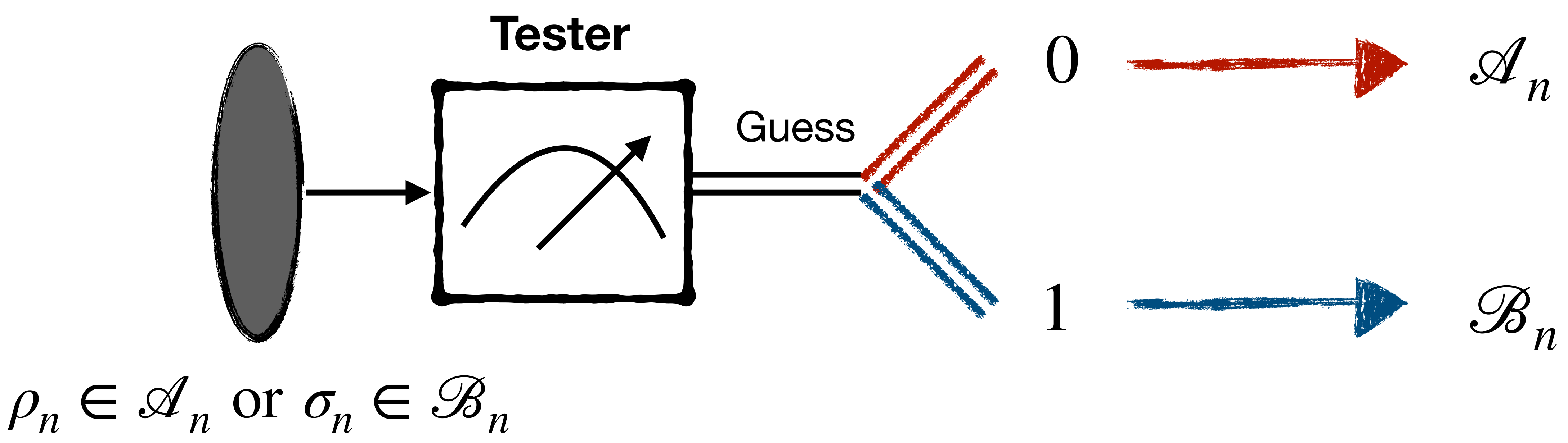
$$\alpha_{n,r}(\rho_n\|\sigma_n) := \min_{0\le M_n\le I}\{\alpha(\rho_n, M_n) : \beta(\sigma_n, M_n) \le 2^{-nr}\}$$

$$\lim_{n\to\infty} -\frac{1}{n}\log \alpha_{n,r}(\rho^{\otimes n}, \sigma^{\otimes n}) = H_r(\rho\|\sigma)$$

Quantum Hoeffding divergence

$$H_r(\rho\|\sigma) := \sup_{\alpha\in(0,1)} \frac{\alpha-1}{\alpha}\big(r - D_{\mathrm{P},\alpha}(\rho\|\sigma)\big)$$

Petz $$D_{\mathrm{P},\alpha}(\rho\|\sigma) := \frac{1}{\alpha-1}\log \mathrm{Tr}\left[\rho^{\alpha}\sigma^{1-\alpha}\right]$$



$$\alpha_{n,r}(\mathscr{A}_n\|\mathscr{B}_n) := \min_{0\le M_n\le I}\{\alpha(\mathscr{A}_n, M_n) : \beta(\mathscr{B}_n, M_n) \le 2^{-nr}\}$$

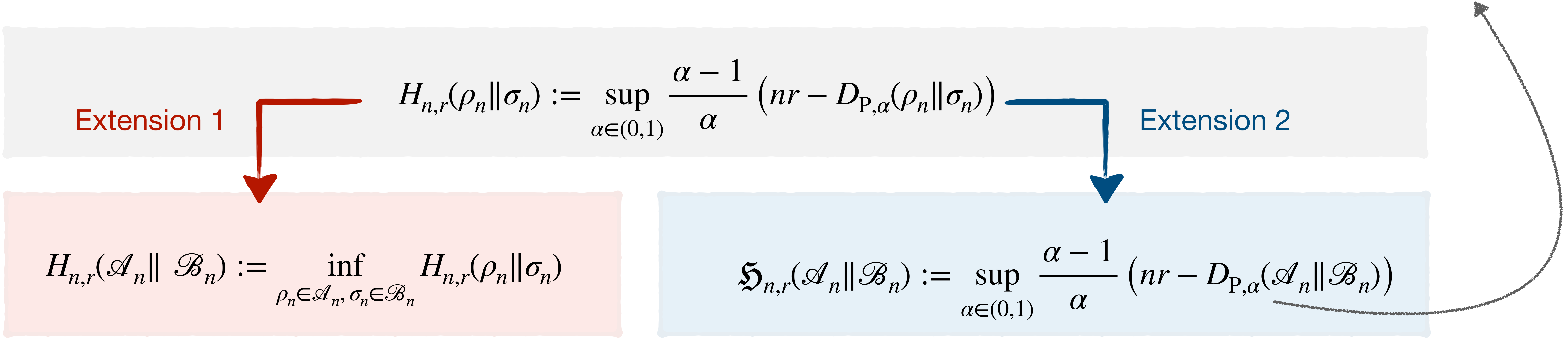
$$\lim_{n\to\infty} -\frac{1}{n}\log \alpha_{n,r}(\mathscr{A}_n, \mathscr{B}_n) \overset{?}{=} H_r(\mathscr{A}\|\mathscr{B})$$

How to define $H_r(\mathscr{A}\|\mathscr{B})$?

(II) Error exponent (2508.12901)

How to define $H_r(\mathscr{A}\|\mathscr{B})$?

$$D_{\mathrm{P},\alpha}(\mathscr{A}_n\|\mathscr{B}_n) := \inf_{\rho_n\in\mathscr{A}_n,\ \sigma_n\in\mathscr{B}_n} D_{\mathrm{P},\alpha}(\rho_n\|\sigma_n)$$



$$H_{n,r}(\rho_n\|\sigma_n) := \sup_{\alpha\in(0,1)} \frac{\alpha-1}{\alpha}\big(nr - D_{\mathrm{P},\alpha}(\rho_n\|\sigma_n)\big)$$

Extension 1

Extension 2

$$H_{n,r}(\mathscr{A}_n\|\ \mathscr{B}_n) := \inf_{\rho_n\in\mathscr{A}_n,\sigma_n\in\mathscr{B}_n} H_{n,r}(\rho_n\|\sigma_n)$$

$$\mathfrak{H}_{n,r}(\mathscr{A}_n\|\mathscr{B}_n) := \sup_{\alpha\in(0,1)} \frac{\alpha-1}{\alpha}\big(nr - D_{\mathrm{P},\alpha}(\mathscr{A}_n\|\mathscr{B}_n)\big)$$

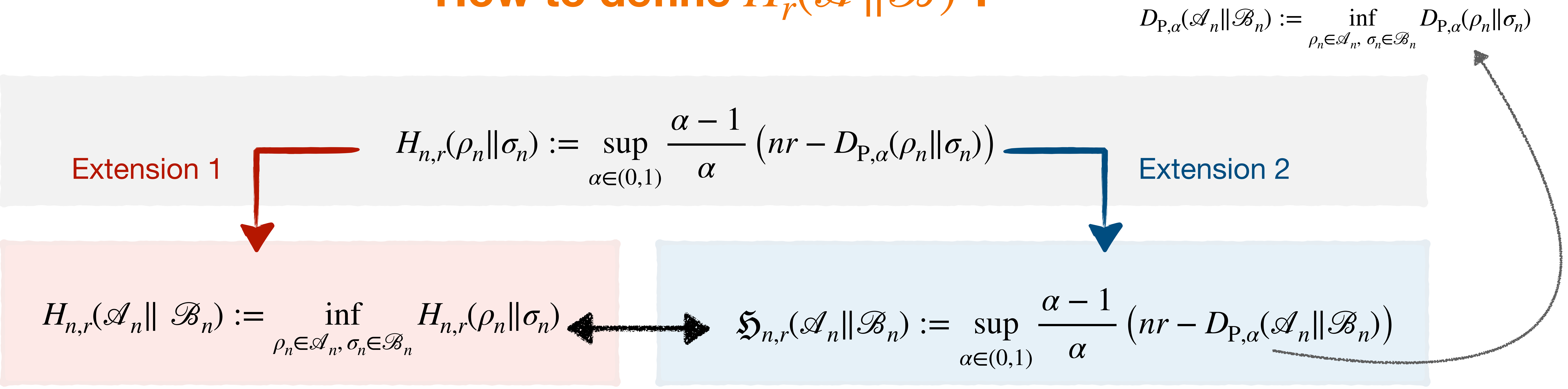
Are they equivalent?

$$\inf_{\rho_n\in\mathscr{A}_n,\sigma_n\in\mathscr{B}_n} \sup_{\alpha\in(0,1)} \underbrace{\frac{\alpha-1}{\alpha}\big(nr - D_{\mathrm{P},\alpha}(\rho_n\|\sigma_n)\big)}_{\text{Convex in }(\rho_n,\sigma_n)} \underset{\text{minimax theorem}}{=\!=\!=} \sup_{\alpha\in(0,1)} \inf_{\rho_n\in\mathscr{A}_n,\sigma_n\in\mathscr{B}_n} \frac{\alpha-1}{\alpha}\big(nr - D_{\mathrm{P},\alpha}(\rho_n\|\sigma_n)\big)$$

Replace $u = (\alpha-1)/\alpha$, then concave in u

(II) Error exponent (2508.12901)

How to define $H_r(\mathscr{A}\|\mathscr{B})$?



$$D_{\mathrm{P},\alpha}(\mathscr{A}_n\|\mathscr{B}_n) := \inf_{\rho_n\in\mathscr{A}_n,\ \sigma_n\in\mathscr{B}_n} D_{\mathrm{P},\alpha}(\rho_n\|\sigma_n)$$

$$H_{n,r}(\rho_n\|\sigma_n) := \sup_{\alpha\in(0,1)} \frac{\alpha-1}{\alpha}\big(nr - D_{\mathrm{P},\alpha}(\rho_n\|\sigma_n)\big)$$

Extension 1

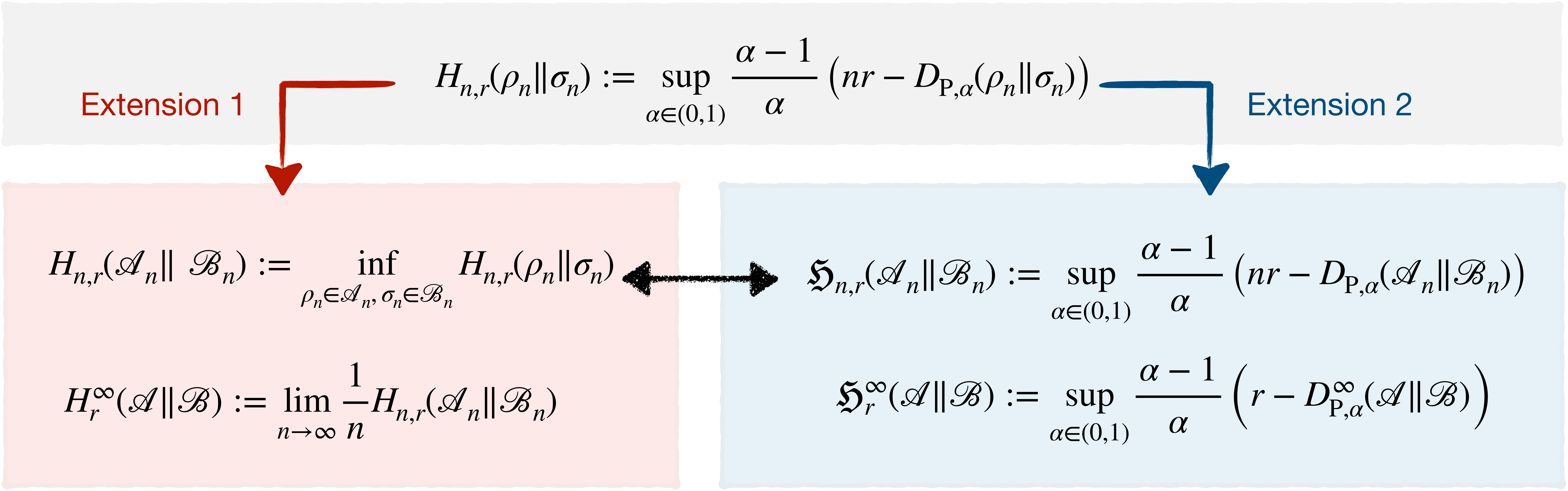
$$H_{n,r}(\mathscr{A}_n\|\ \mathscr{B}_n) := \inf_{\rho_n\in\mathscr{A}_n,\sigma_n\in\mathscr{B}_n} H_{n,r}(\rho_n\|\sigma_n)$$

Extension 2

$$\mathfrak{H}_{n,r}(\mathscr{A}_n\|\mathscr{B}_n) := \sup_{\alpha\in(0,1)} \frac{\alpha-1}{\alpha}\big(nr - D_{\mathrm{P},\alpha}(\mathscr{A}_n\|\mathscr{B}_n)\big)$$

(II) Error exponent (2508.12901)

How to define $H_r(\mathscr{A}\|\mathscr{B})$?



$$H_{n,r}(\rho_n\|\sigma_n) := \sup_{\alpha\in(0,1)} \frac{\alpha-1}{\alpha}\Big(nr - D_{\mathrm{P},\alpha}(\rho_n\|\sigma_n)\Big)$$

Extension 1

Extension 2

$$H_{n,r}(\mathscr{A}_n\|\ \mathscr{B}_n) := \inf_{\rho_n\in\mathscr{A}_n,\, \sigma_n\in\mathscr{B}_n} H_{n,r}(\rho_n\|\sigma_n) \longleftrightarrow \mathfrak{H}_{n,r}(\mathscr{A}_n\|\mathscr{B}_n) := \sup_{\alpha\in(0,1)} \frac{\alpha-1}{\alpha}\Big(nr - D_{\mathrm{P},\alpha}(\mathscr{A}_n\|\mathscr{B}_n)\Big)$$

$$H_r^{\infty}(\mathscr{A}\|\mathscr{B}) := \lim_{n\to\infty} \frac{1}{n} H_{n,r}(\mathscr{A}_n\|\mathscr{B}_n)$$

$$\mathfrak{H}_r^{\infty}(\mathscr{A}\|\mathscr{B}) := \sup_{\alpha\in(0,1)} \frac{\alpha-1}{\alpha}\Big(r - D_{\mathrm{P},\alpha}^{\infty}(\mathscr{A}\|\mathscr{B})\Big)$$

Are they equivalent?

$$\inf_{n\geq 1} \sup_{\alpha\in(0,1)} \frac{\alpha-1}{\alpha}\left(r - \frac{1}{n} D_{\mathrm{P},\alpha}(\mathscr{A}_n\|\mathscr{B}_n)\right) \geq \sup_{\alpha\in(0,1)} \inf_{n\geq 1} \frac{\alpha-1}{\alpha}\left(r - \frac{1}{n} D_{\mathrm{P},\alpha}(\mathscr{A}_n\|\mathscr{B}_n)\right)$$

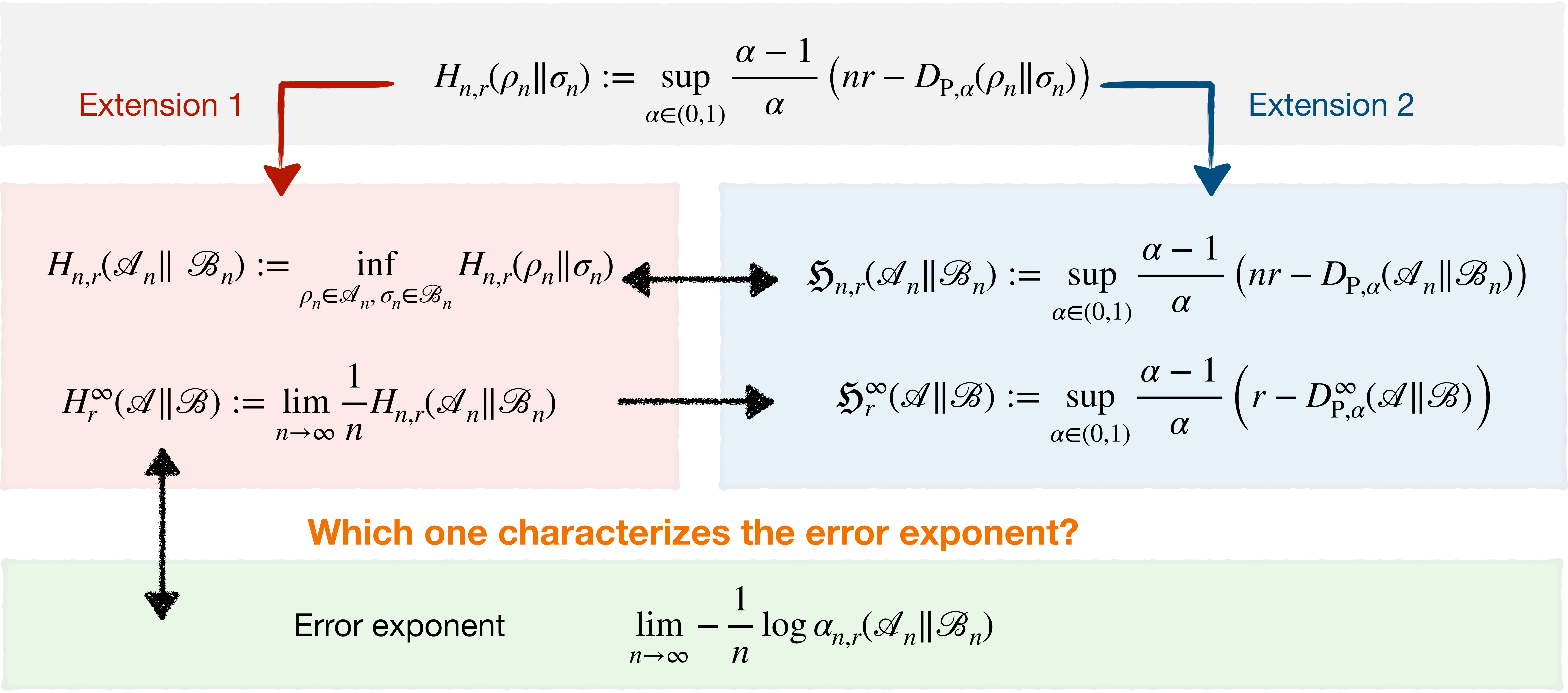
minimax inequality

Challenge for equality: both sets are non-compact

Discontinuity example for regularized Petz divergence

(II) Error exponent (2508.12901)

How to define $H_r(\mathscr{A}\|\mathscr{B})$?



$$H_{n,r}(\rho_n\|\sigma_n) := \sup_{\alpha\in(0,1)} \frac{\alpha-1}{\alpha}\left(nr - D_{\mathrm{P},\alpha}(\rho_n\|\sigma_n)\right)$$

Extension 1

Extension 2

$$H_{n,r}(\mathscr{A}_n\|\mathscr{B}_n) := \inf_{\rho_n\in\mathscr{A}_n,\,\sigma_n\in\mathscr{B}_n} H_{n,r}(\rho_n\|\sigma_n)$$

$$H_r^{\infty}(\mathscr{A}\|\mathscr{B}) := \lim_{n\to\infty}\frac{1}{n}H_{n,r}(\mathscr{A}_n\|\mathscr{B}_n)$$

$$\mathfrak{H}_{n,r}(\mathscr{A}_n\|\mathscr{B}_n) := \sup_{\alpha\in(0,1)} \frac{\alpha-1}{\alpha}\left(nr - D_{\mathrm{P},\alpha}(\mathscr{A}_n\|\mathscr{B}_n)\right)$$

$$\mathfrak{H}_r^{\infty}(\mathscr{A}\|\mathscr{B}) := \sup_{\alpha\in(0,1)} \frac{\alpha-1}{\alpha}\left(r - D^{\infty}_{\mathrm{P},\alpha}(\mathscr{A}\|\mathscr{B})\right)$$

Which one characterizes the error exponent?

Error exponent $\lim_{n\to\infty} -\frac{1}{n}\log \alpha_{n,r}(\mathscr{A}_n\|\mathscr{B}_n)$

(II) Error exponent (2508.12901)

Error exponent $$\underline{\lim}_{n\to\infty} -\frac{1}{n}\log \alpha_{n,r}(\mathscr{A}_n\|\mathscr{B}_n) = H_r^{\infty}(\mathscr{A}\|\mathscr{B})$$

Proof of the lower bound: $\mathrm{Tr}[V^{\alpha}W^{1-\alpha}] \geq \mathrm{Tr}W\{W \leq V\} + \mathrm{Tr}V\{W > V\}$ [Audenaert et al.-07]

Proof of the upper bound:

$$\underline{\lim}_{n\to\infty} -\frac{1}{n}\log \alpha_{n,r}(\mathscr{A}_n\|\mathscr{B}_n) \leq \underline{\lim}_{n\to\infty} -\frac{1}{mn}\log \alpha_{mn,r}(\mathscr{A}_{mn}, \mathscr{B}_{mn})$$

Property of lower limit

Minimax theorem

$$= \underline{\lim}_{n\to\infty} -\frac{1}{mn}\log \sup_{\substack{\rho_{mn}\in\mathscr{A}_{mn}\\ \sigma_{mn}\in\mathscr{B}_{mn}}} \alpha_{mn,r}(\rho_{mn}, \sigma_{mn})$$

Stability assumption

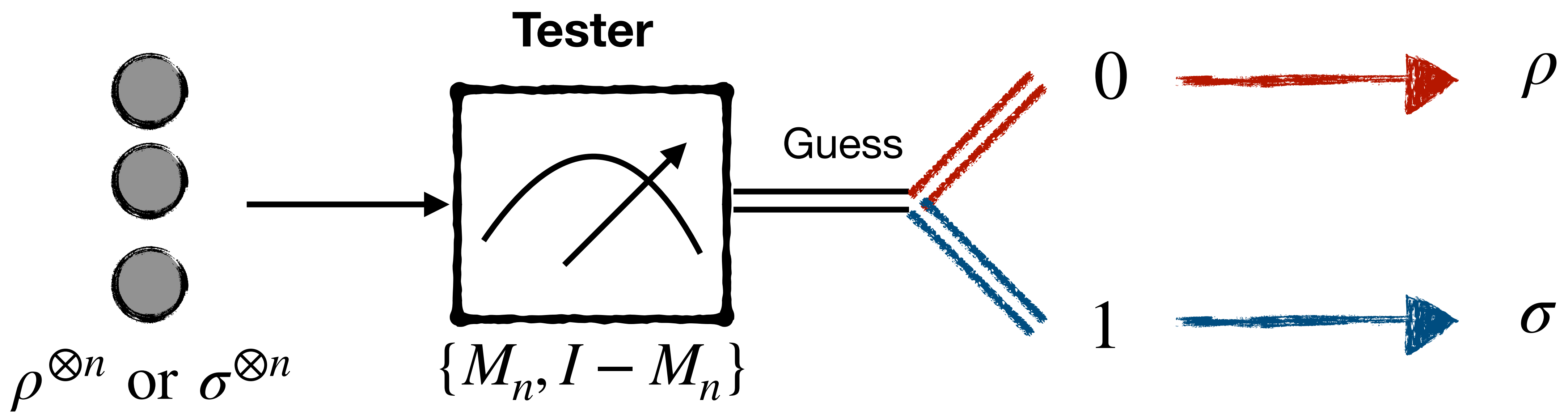
$$\leq \underline{\lim}_{n\to\infty} -\frac{1}{mn}\log \alpha_{mn,r}(\rho_m^{\otimes n}, \sigma_m^{\otimes n})$$

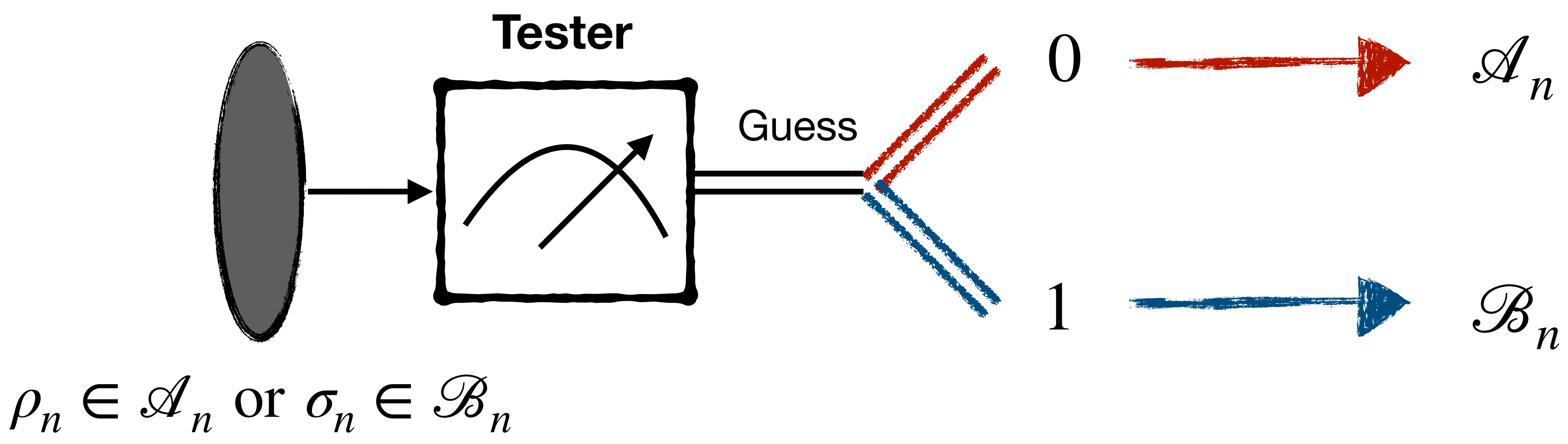
Hoeffding for i.i.d. states

$$= \frac{1}{m}H_{m,r}(\rho_m\|\sigma_m) \longrightarrow \frac{1}{m}H_{m,r}(\mathscr{A}_m\|\mathscr{B}_m) \longrightarrow H_r^{\infty}(\mathscr{A}\|\mathscr{B})$$

To show the upper limit for the upper bound, we can construct sequence from a subsequence
This requires Nussbaum-Szkoła distributions and the Gartner-Ellis theorem; refer to the full paper

(III) Strong converse exponent (2508.12901)





$$\alpha_{n,r}(\rho_n\|\sigma_n) := \min_{0\le M_n\le I}\{\alpha(\rho_n, M_n) : \beta(\sigma_n, M_n) \le 2^{-nr}\}$$

$$\lim_{n\to\infty} -\frac{1}{n}\log(1-\alpha_{n,r}(\rho^{\otimes n}, \sigma^{\otimes n})) = H_r^*(\rho\|\sigma)$$

Quantum Hoeffding anti-divergence

$$H_r^*(\rho\|\sigma) := \sup_{\alpha>1} \frac{\alpha-1}{\alpha}\left(r - D_{\mathrm{S},\alpha}(\rho\|\sigma)\right)$$

Sandwiched $$D_{\mathrm{S},\alpha}(\rho\|\sigma) := \frac{1}{\alpha-1}\log \operatorname{Tr}\left[\sigma^{\frac{1-\alpha}{2\alpha}}\rho\,\sigma^{\frac{1-\alpha}{2\alpha}}\right]^{\alpha}$$

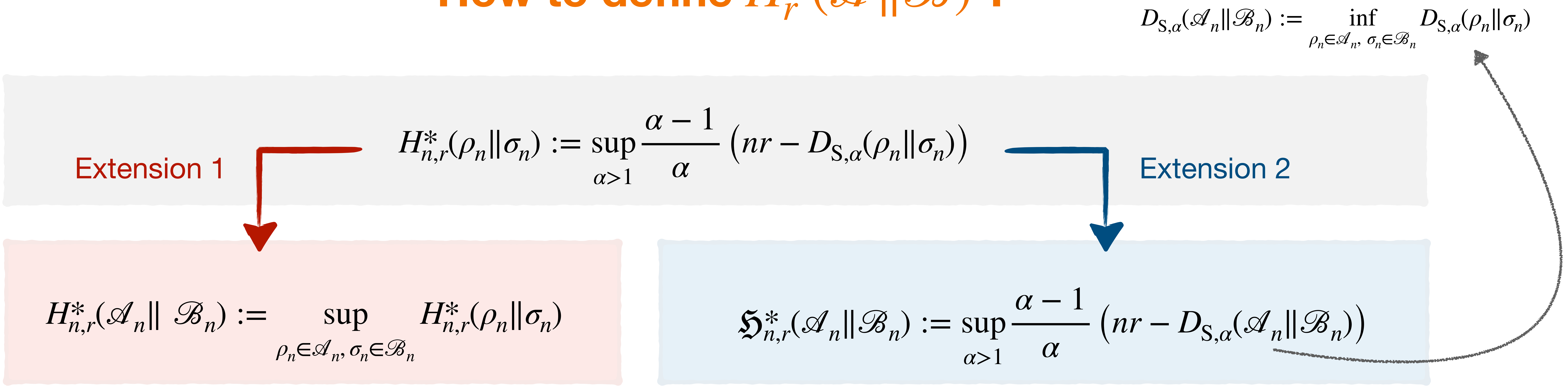
$$\alpha_{n,r}(\mathscr{A}_n\|\mathscr{B}_n) := \min_{0\le M_n\le I}\{\alpha(\mathscr{A}_n, M_n) : \beta(\mathscr{B}_n, M_n) \le 2^{-nr}\}$$

$$\lim_{n\to\infty} -\frac{1}{n}\log(1-\alpha_{n,r}(\mathscr{A}_n, \mathscr{B}_n)) \overset{?}{=} H_r^*(\mathscr{A}\|\mathscr{B})$$

How to define $H_r^*(\mathscr{A}\|\mathscr{B})$?

(III) Strong converse exponent (2508.12901)

How to define $H^*_r(\mathscr{A}\|\mathscr{B})$?



$$D_{\mathrm{S},\alpha}(\mathscr{A}_n\|\mathscr{B}_n) := \inf_{\rho_n\in\mathscr{A}_n,\ \sigma_n\in\mathscr{B}_n} D_{\mathrm{S},\alpha}(\rho_n\|\sigma_n)$$

$$H^*_{n,r}(\rho_n\|\sigma_n) := \sup_{\alpha>1}\frac{\alpha-1}{\alpha}\big(nr - D_{\mathrm{S},\alpha}(\rho_n\|\sigma_n)\big)$$

Extension 1

$$H^*_{n,r}(\mathscr{A}_n\|\mathscr{B}_n) := \sup_{\rho_n\in\mathscr{A}_n,\,\sigma_n\in\mathscr{B}_n} H^*_{n,r}(\rho_n\|\sigma_n)$$

Extension 2

$$\mathfrak{H}^*_{n,r}(\mathscr{A}_n\|\mathscr{B}_n) := \sup_{\alpha>1}\frac{\alpha-1}{\alpha}\big(nr - D_{\mathrm{S},\alpha}(\mathscr{A}_n\|\mathscr{B}_n)\big)$$

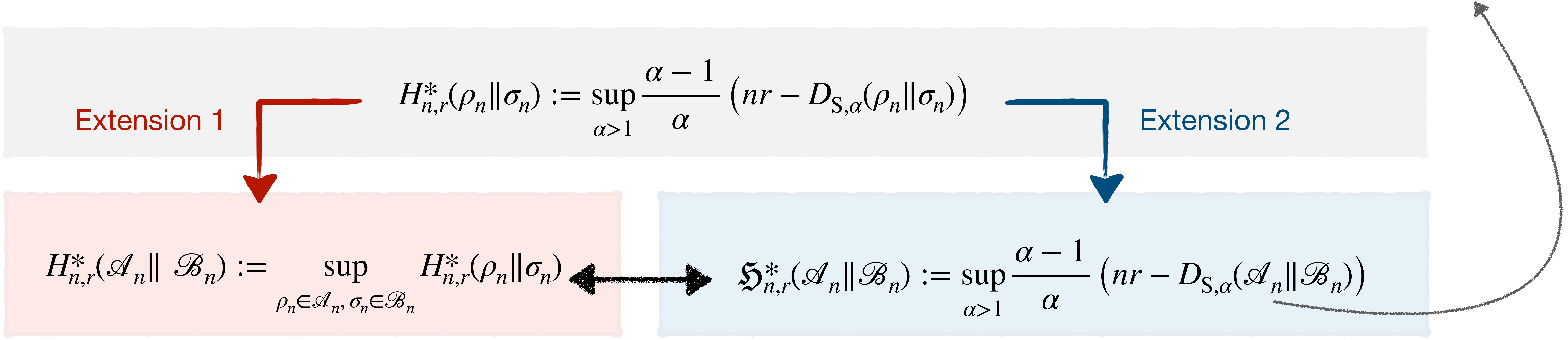
Are they equivalent?

$$\sup_{\rho_n\in\mathscr{A}_n,\,\sigma_n\in\mathscr{B}_n}\ \sup_{\alpha>1}\frac{\alpha-1}{\alpha}\big(nr - D_{\mathrm{S},\alpha}(\rho_n\|\sigma_n)\big) = \sup_{\alpha>1}\ \sup_{\rho_n\in\mathscr{A}_n,\,\sigma_n\in\mathscr{B}_n}\frac{\alpha-1}{\alpha}\big(nr - D_{\mathrm{S},\alpha}(\rho_n\|\sigma_n)\big)$$

No minimax issue here

(III) Strong converse exponent (2508.12901)

How to define $H^*_r(\mathscr{A}\|\mathscr{B})$?



$$D_{\mathrm{S},\alpha}(\mathscr{A}_n\|\mathscr{B}_n) := \inf_{\rho_n\in\mathscr{A}_n,\ \sigma_n\in\mathscr{B}_n} D_{\mathrm{S},\alpha}(\rho_n\|\sigma_n)$$

$$H^*_{n,r}(\rho_n\|\sigma_n) := \sup_{\alpha>1}\frac{\alpha-1}{\alpha}\big(nr - D_{\mathrm{S},\alpha}(\rho_n\|\sigma_n)\big)$$

Extension 1

Extension 2

$$H^*_{n,r}(\mathscr{A}_n\|\ \mathscr{B}_n) := \sup_{\rho_n\in\mathscr{A}_n,\,\sigma_n\in\mathscr{B}_n} H^*_{n,r}(\rho_n\|\sigma_n)$$

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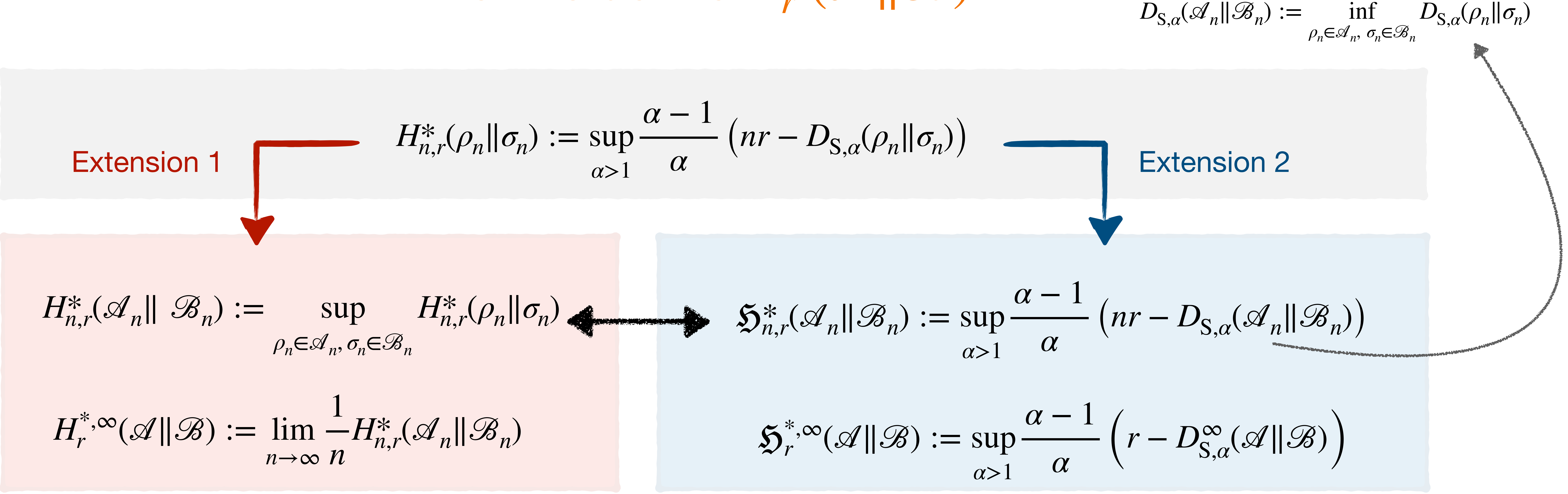
(III) Strong converse exponent (2508.12901)

How to define $H_r^*(\mathscr{A}\|\mathscr{B})$?

$$D_{\mathrm{S},\alpha}(\mathscr{A}_n\|\mathscr{B}_n) := \inf_{\rho_n\in\mathscr{A}_n,\ \sigma_n\in\mathscr{B}_n} D_{\mathrm{S},\alpha}(\rho_n\|\sigma_n)$$



$$H^*_{n,r}(\rho_n\|\sigma_n) := \sup_{\alpha>1} \frac{\alpha-1}{\alpha}\left(nr - D_{\mathrm{S},\alpha}(\rho_n\|\sigma_n)\right)$$

Extension 1

Extension 2

$$H^*_{n,r}(\mathscr{A}_n\|\ \mathscr{B}_n) := \sup_{\rho_n\in\mathscr{A}_n,\,\sigma_n\in\mathscr{B}_n} H^*_{n,r}(\rho_n\|\sigma_n)$$

$$H_r^{*,\infty}(\mathscr{A}\|\mathscr{B}) := \lim_{n\to\infty}\frac{1}{n}H^*_{n,r}(\mathscr{A}_n\|\mathscr{B}_n)$$

$$\mathfrak{H}^*_{n,r}(\mathscr{A}_n\|\mathscr{B}_n) := \sup_{\alpha>1}\frac{\alpha-1}{\alpha}\left(nr - D_{\mathrm{S},\alpha}(\mathscr{A}_n\|\mathscr{B}_n)\right)$$

$$\mathfrak{H}_r^{*,\infty}(\mathscr{A}\|\mathscr{B}) := \sup_{\alpha>1}\frac{\alpha-1}{\alpha}\left(r - D^{\infty}_{\mathrm{S},\alpha}(\mathscr{A}\|\mathscr{B})\right)$$

Are they equivalent?

$$\underset{\substack{\Updownarrow\\ \sup_{n>1}}}{\lim_{n\to\infty}}\ \sup_{\alpha>1}\frac{\alpha-1}{\alpha}\left(r-\frac{1}{n}D_{\mathrm{S},\alpha}(\mathscr{A}_n\|\mathscr{B}_n)\right)$$

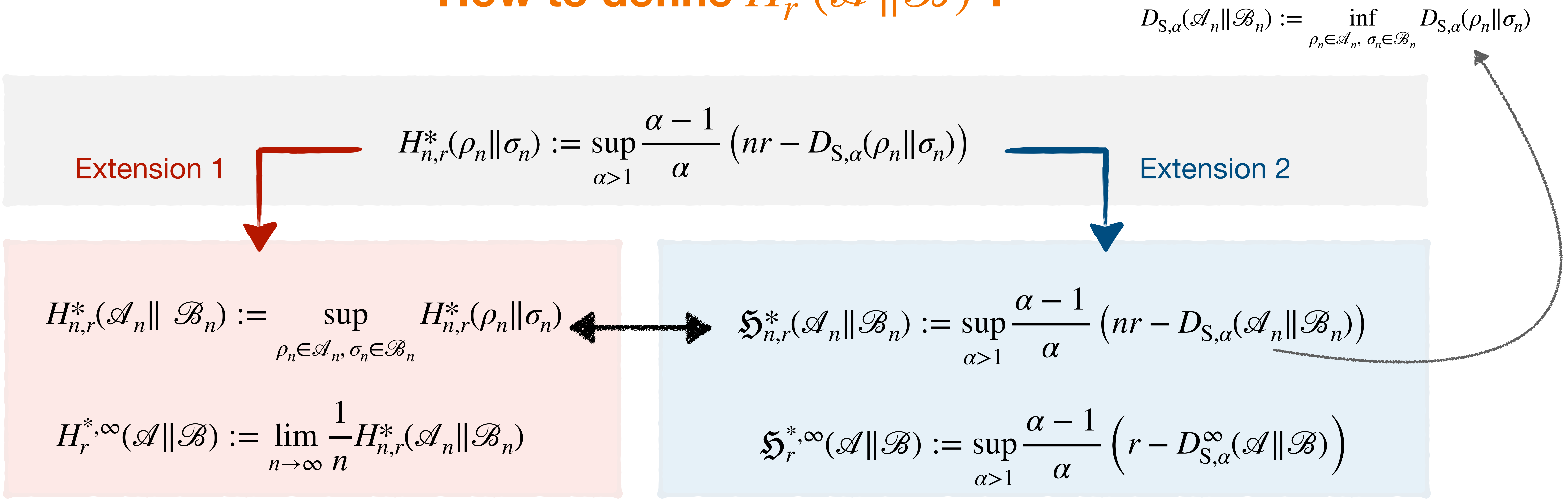
Superadditivity in n

$$\sup_{\alpha>1}\ \underset{\substack{\Updownarrow\\ \sup_{n>1}}}{\lim_{n\to\infty}}\frac{\alpha-1}{\alpha}\left(r-\frac{1}{n}D_{\mathrm{S},\alpha}(\mathscr{A}_n\|\mathscr{B}_n)\right)$$

Superadditivity in n

(III) Strong converse exponent (2508.12901)

How to define $H_r^*(\mathscr{A}\|\mathscr{B})$?



$$D_{\mathrm{S},\alpha}(\mathscr{A}_n\|\mathscr{B}_n) := \inf_{\rho_n\in\mathscr{A}_n,\ \sigma_n\in\mathscr{B}_n} D_{\mathrm{S},\alpha}(\rho_n\|\sigma_n)$$

$$H^*_{n,r}(\rho_n\|\sigma_n) := \sup_{\alpha>1}\frac{\alpha-1}{\alpha}\left(nr - D_{\mathrm{S},\alpha}(\rho_n\|\sigma_n)\right)$$

Extension 1

Extension 2

$$H^*_{n,r}(\mathscr{A}_n\|\mathscr{B}_n) := \sup_{\rho_n\in\mathscr{A}_n,\,\sigma_n\in\mathscr{B}_n} H^*_{n,r}(\rho_n\|\sigma_n)$$

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$$\mathfrak{H}^*_{n,r}(\mathscr{A}_n\|\mathscr{B}_n) := \sup_{\alpha>1}\frac{\alpha-1}{\alpha}\left(nr - D_{\mathrm{S},\alpha}(\mathscr{A}_n\|\mathscr{B}_n)\right)$$

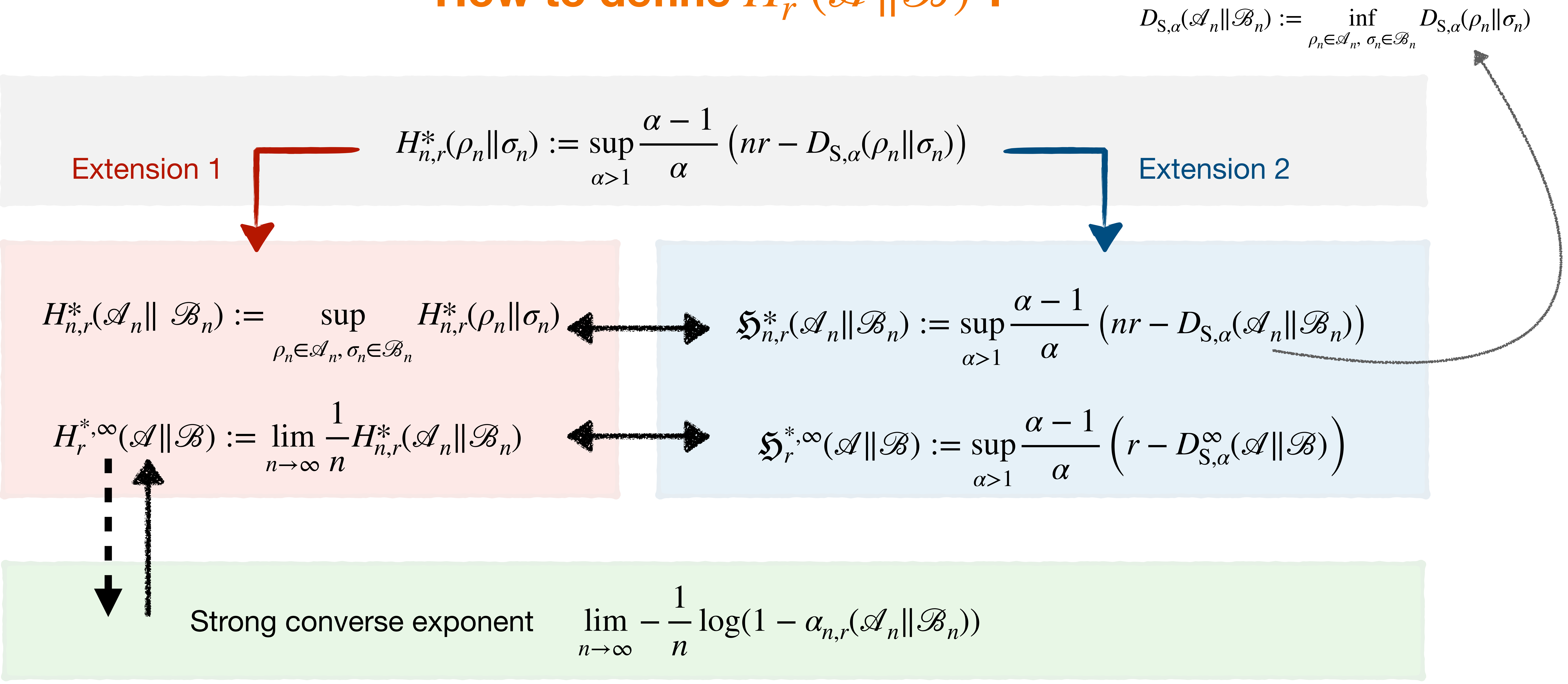
$$\mathfrak{H}_r^{*,\infty}(\mathscr{A}\|\mathscr{B}) := \sup_{\alpha>1}\frac{\alpha-1}{\alpha}\left(r - D^{\infty}_{\mathrm{S},\alpha}(\mathscr{A}\|\mathscr{B})\right)$$

Are they equivalent?

$$\begin{array}{c}\displaystyle\lim_{n\to\infty}\\ \updownarrow\\ \displaystyle\sup_{n>1}\end{array}\ \sup_{\alpha>1}\underbrace{\frac{\alpha-1}{\alpha}\left(r-\frac{1}{n}D_{\mathrm{S},\alpha}(\mathscr{A}_n\|\mathscr{B}_n)\right)}_{\text{Superadditivity in } n} = \sup_{\alpha>1}\ \begin{array}{c}\displaystyle\lim_{n\to\infty}\\ \updownarrow\\ \displaystyle\sup_{n>1}\end{array}\underbrace{\frac{\alpha-1}{\alpha}\left(r-\frac{1}{n}D_{\mathrm{S},\alpha}(\mathscr{A}_n\|\mathscr{B}_n)\right)}_{\text{Superadditivity in } n}$$

(III) Strong converse exponent (2508.12901)

How to define $H_r^*(\mathscr{A}\|\mathscr{B})$?



$$D_{\mathrm{S},\alpha}(\mathscr{A}_n\|\mathscr{B}_n) := \inf_{\rho_n\in\mathscr{A}_n,\ \sigma_n\in\mathscr{B}_n} D_{\mathrm{S},\alpha}(\rho_n\|\sigma_n)$$

$$H^*_{n,r}(\rho_n\|\sigma_n) := \sup_{\alpha>1}\frac{\alpha-1}{\alpha}\left(nr - D_{\mathrm{S},\alpha}(\rho_n\|\sigma_n)\right)$$

Extension 1

Extension 2

$$H^*_{n,r}(\mathscr{A}_n\|\mathscr{B}_n) := \sup_{\rho_n\in\mathscr{A}_n,\,\sigma_n\in\mathscr{B}_n} H^*_{n,r}(\rho_n\|\sigma_n)$$

$$H_r^{*,\infty}(\mathscr{A}\|\mathscr{B}) := \lim_{n\to\infty}\frac{1}{n}H^*_{n,r}(\mathscr{A}_n\|\mathscr{B}_n)$$

$$\mathfrak{H}^*_{n,r}(\mathscr{A}_n\|\mathscr{B}_n) := \sup_{\alpha>1}\frac{\alpha-1}{\alpha}\left(nr - D_{\mathrm{S},\alpha}(\mathscr{A}_n\|\mathscr{B}_n)\right)$$

$$\mathfrak{H}_r^{*,\infty}(\mathscr{A}\|\mathscr{B}) := \sup_{\alpha>1}\frac{\alpha-1}{\alpha}\left(r - D^{\infty}_{\mathrm{S},\alpha}(\mathscr{A}\|\mathscr{B})\right)$$

Strong converse exponent $\displaystyle\lim_{n\to\infty} -\frac{1}{n}\log(1-\alpha_{n,r}(\mathscr{A}_n\|\mathscr{B}_n))$

The dashed arrow indicates partial progress when $\mathscr{A}_n$ is a singleton and additional assumptions.

Refinement of the Stein exponent regime

Error exponent

$$\liminf_{n\to\infty} -\frac{1}{n}\log \alpha_{n,r}(\mathscr{A}_n\|\mathscr{B}_n) \geq \mathfrak{H}_r^{\infty}(\mathscr{A}\|\mathscr{B})$$

Strong converse exponent

$$\liminf_{n\to\infty} -\frac{1}{n}\log(1-\alpha_{n,r}(\mathscr{A}_n\|\mathscr{B}_n)) \geq \mathfrak{H}_r^{*,\infty}(\mathscr{A}\|\mathscr{B})$$

$$\mathfrak{H}_r^{\infty}(\mathscr{A}\|\mathscr{B}) := \sup_{\alpha\in(0,1)} \frac{\alpha-1}{\alpha}\left(r - D_{\mathrm{P},\alpha}^{\infty}(\mathscr{A}\|\mathscr{B})\right)$$

$$\mathfrak{H}_r^{*,\infty}(\mathscr{A}\|\mathscr{B}) := \sup_{\alpha>1} \frac{\alpha-1}{\alpha}\left(r - D_{\mathrm{S},\alpha}^{\infty}(\mathscr{A}\|\mathscr{B})\right)$$

Refinement of the Stein exponent regime

Error exponent

$$\liminf_{n\to\infty} -\frac{1}{n}\log \alpha_{n,r}(\mathscr{A}_n\|\mathscr{B}_n) \geq \mathfrak{H}_r^{\infty}(\mathscr{A}\|\mathscr{B})$$

Strong converse exponent

$$\liminf_{n\to\infty} -\frac{1}{n}\log(1-\alpha_{n,r}(\mathscr{A}_n\|\mathscr{B}_n)) \geq \mathfrak{H}_r^{*,\infty}(\mathscr{A}\|\mathscr{B})$$

$$\mathfrak{H}_r^{\infty}(\mathscr{A}\|\mathscr{B}) := \sup_{\alpha\in(0,1)} \frac{\alpha-1}{\alpha}\left(r - D_{\mathrm{P},\alpha}^{\infty}(\mathscr{A}\|\mathscr{B})\right)$$

$$\mathfrak{H}_r^{*,\infty}(\mathscr{A}\|\mathscr{B}) := \sup_{\alpha>1} \frac{\alpha-1}{\alpha}\left(r - D_{\mathrm{S},\alpha}^{\infty}(\mathscr{A}\|\mathscr{B})\right)$$

Using the same assumptions in **[KF-Fawzi-Fawzi-24]**

$$\sup_{\alpha\in(0,1)} D_{\mathrm{P},\alpha}^{\infty}(\mathscr{A}\|\mathscr{B}) = D^{\infty}(\mathscr{A}\|\mathscr{B}) = \inf_{\alpha>1} D_{\mathrm{S},\alpha}^{\infty}(\mathscr{A}\|\mathscr{B})$$

Refinement of the Stein exponent regime

Error exponent

$$\liminf_{n\to\infty} -\frac{1}{n}\log \alpha_{n,r}(\mathscr{A}_n\|\mathscr{B}_n) \geq \mathfrak{H}_r^{\infty}(\mathscr{A}\|\mathscr{B}) > 0, \quad \forall\ 0 < r < D^{\infty}(\mathscr{A}\|\mathscr{B})$$

Strong converse exponent

$$\liminf_{n\to\infty} -\frac{1}{n}\log(1-\alpha_{n,r}(\mathscr{A}_n\|\mathscr{B}_n)) \geq \mathfrak{H}_r^{*,\infty}(\mathscr{A}\|\mathscr{B}) > 0, \quad \forall\ r > D^{\infty}(\mathscr{A}\|\mathscr{B})$$

$$\mathfrak{H}_r^{\infty}(\mathscr{A}\|\mathscr{B}) := \sup_{\alpha\in(0,1)} \frac{\alpha-1}{\alpha}\left(r - D_{\mathrm{P},\alpha}^{\infty}(\mathscr{A}\|\mathscr{B})\right)$$

$$\mathfrak{H}_r^{*,\infty}(\mathscr{A}\|\mathscr{B}) := \sup_{\alpha>1} \frac{\alpha-1}{\alpha}\left(r - D_{\mathrm{S},\alpha}^{\infty}(\mathscr{A}\|\mathscr{B})\right)$$

Using the same assumptions in **[KF-Fawzi-Fawzi-24]**

$$\sup_{\alpha\in(0,1)} D_{\mathrm{P},\alpha}^{\infty}(\mathscr{A}\|\mathscr{B}) = D^{\infty}(\mathscr{A}\|\mathscr{B}) = \inf_{\alpha>1} D_{\mathrm{S},\alpha}^{\infty}(\mathscr{A}\|\mathscr{B})$$

If the type-II error exponent is a bit off the Stein's exponent $D^{\infty}(\mathscr{A}\|\mathscr{B})$ then the type-I error will exponentially decays to zero or one. So $D^{\infty}(\mathscr{A}\|\mathscr{B})$ is a sharp threshold.

Refinement of the Stein exponent regime

Error exponent

$$\liminf_{n\to\infty} -\frac{1}{n}\log \alpha_{n,r}(\mathscr{A}_n\|\mathscr{B}_n) \ge \mathfrak{H}_r^{\infty}(\mathscr{A}\|\mathscr{B}) > 0, \quad \forall\ 0 < r < D^{\infty}(\mathscr{A}\|\mathscr{B})$$

Strong converse exponent

$$\liminf_{n\to\infty} -\frac{1}{n}\log(1-\alpha_{n,r}(\mathscr{A}_n\|\mathscr{B}_n)) \ge \mathfrak{H}_r^{*,\infty}(\mathscr{A}\|\mathscr{B}) > 0, \quad \forall\ r > D^{\infty}(\mathscr{A}\|\mathscr{B})$$

$$\mathfrak{H}_r^{\infty}(\mathscr{A}\|\mathscr{B}) := \sup_{\alpha\in(0,1)} \frac{\alpha-1}{\alpha}\left(r - D_{\mathrm{P},\alpha}^{\infty}(\mathscr{A}\|\mathscr{B})\right)$$

$$\mathfrak{H}_r^{*,\infty}(\mathscr{A}\|\mathscr{B}) := \sup_{\alpha>1} \frac{\alpha-1}{\alpha}\left(r - D_{\mathrm{S},\alpha}^{\infty}(\mathscr{A}\|\mathscr{B})\right)$$

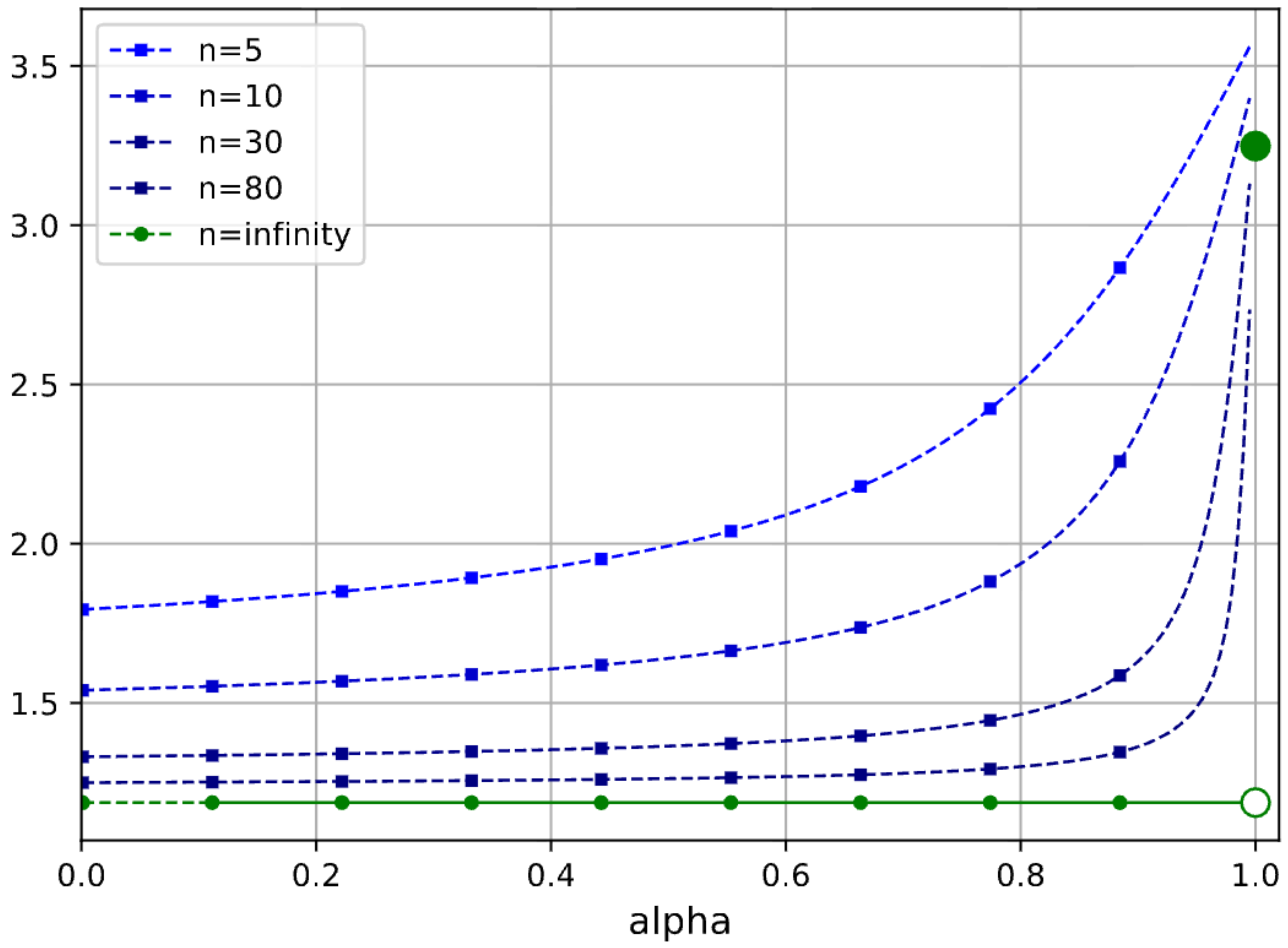
Using the same assumptions in **[KF-Fawzi-Fawzi-24]**

$$\sup_{\alpha\in(0,1)} D_{\mathrm{P},\alpha}^{\infty}(\mathscr{A}\|\mathscr{B}) = D^{\infty}(\mathscr{A}\|\mathscr{B}) = \inf_{\alpha>1} D_{\mathrm{S},\alpha}^{\infty}(\mathscr{A}\|\mathscr{B})$$

Q: Recovering the Stein's setting considered in [Brandao-Plenio-10, Hayashi-Yamasaki-24; Lami-24]?

A: We need $\sup_{\alpha\in(0,1)} D_{\mathrm{P},\alpha}^{\infty}(\mathscr{A}\|\mathscr{B}) = D^{\infty}(\mathscr{A}\|\mathscr{B})$ with the corresponding assumptions.

Examples for discontinuity of regularized divergence



convex compact and stable under tensor product

$$\sup_{\alpha\in(0,1)} D_{\mathrm{P},\alpha}(\mathscr{A}_n\|\mathscr{B}_n) = D(\mathscr{A}_n\|\mathscr{B}_n) \quad \forall n \in \mathbb{N}$$

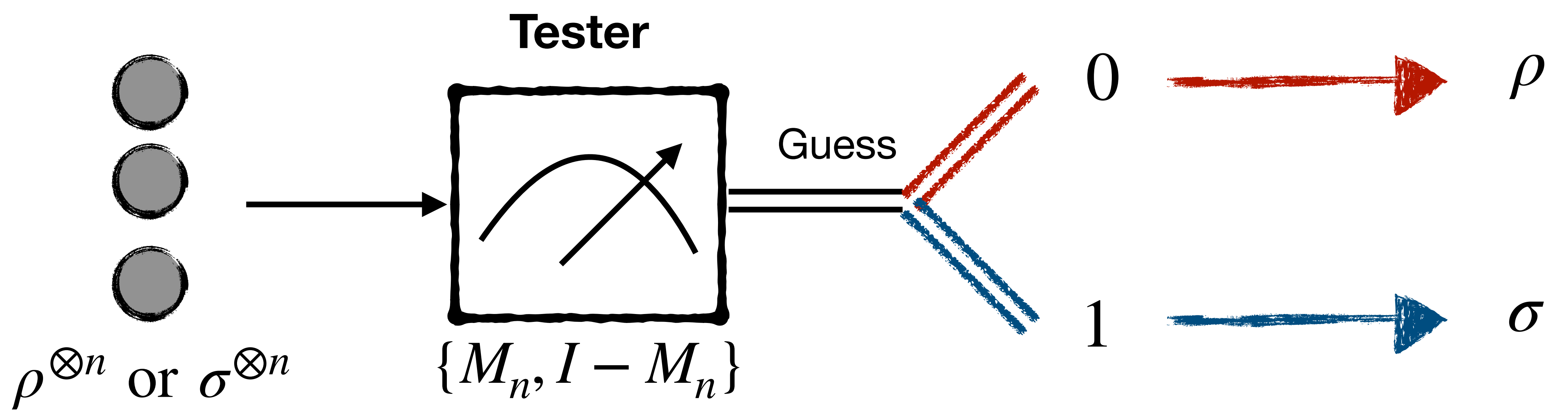
But $\sup_{\alpha\in(0,1)} D^{\infty}_{\mathrm{P},\alpha}(\mathscr{A}_n\|\mathscr{B}_n) \neq D^{\infty}(\mathscr{A}_n\|\mathscr{B}_n)$

Error tradeoff for composite correlated hypotheses is much more complicated than the simple i.i.d. case.

Q: Recovering the Stein's setting considered in [Brandao-Plenio-10, Hayashi-Yamasaki-24; Lami-24]?

A: We need $\sup_{\alpha\in(0,1)} D^{\infty}_{\mathrm{P},\alpha}(\mathscr{A}\|\mathscr{B}) = D^{\infty}(\mathscr{A}\|\mathscr{B})$ with the corresponding assumptions.

(IV) Chernoff exponent (2508.12889)



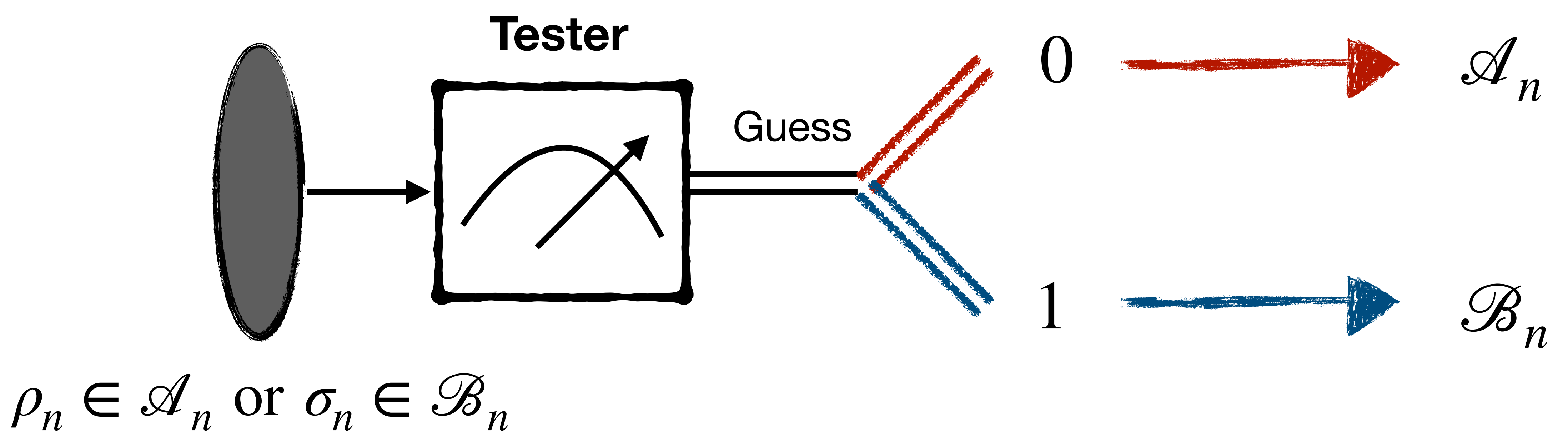
$$P_{e,\min}(\rho_n\|\sigma_n) := \min_{0\le M_n\le I} \left(\pi_0\,\alpha(\rho_n, M_n)+\pi_1\,\beta(\sigma_n, M_n)\right)$$

$$\lim_{n\to\infty} -\frac{1}{n}\log P_{e,\min}(\rho^{\otimes n}, \sigma^{\otimes n}) = C(\rho\|\sigma)$$

Quantum Chernoff divergence

$$C(\rho\|\sigma) := \max_{\alpha\in[0,1]} -\log Q_\alpha(\rho\|\sigma)$$

Petz-quasi $Q_\alpha(\rho\|\sigma) := \mathrm{Tr}\left[\rho^\alpha\sigma^{1-\alpha}\right]$



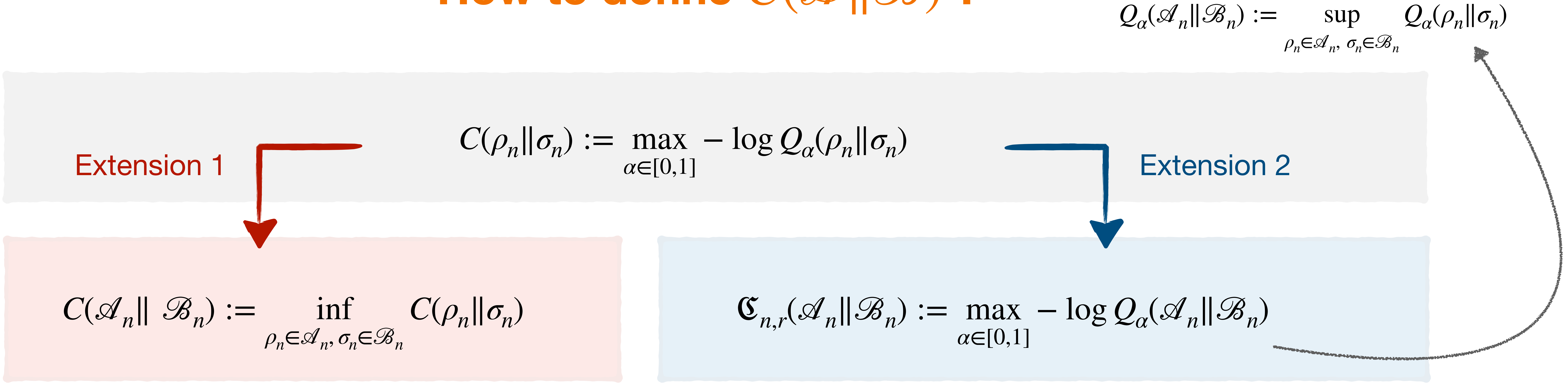
$$P_{e,\min}(\mathscr{A}_n\|\mathscr{B}_n) := \min_{0\le M_n\le I} \left(\pi_0\,\alpha(\mathscr{A}_n, M_n)+\pi_1\,\beta(\mathscr{B}_n, M_n)\right)$$

$$\lim_{n\to\infty} -\frac{1}{n}\log P_{e,\min}(\mathscr{A}_n, \mathscr{B}_n) \overset{?}{=} C(\mathscr{A}\|\mathscr{B})$$

How to define $C(\mathscr{A}\|\mathscr{B})$?

(IV) Chernoff exponent (2508.12889)

How to define $C(\mathscr{A}\|\mathscr{B})$?



$$Q_\alpha(\mathscr{A}_n\|\mathscr{B}_n) := \sup_{\rho_n\in\mathscr{A}_n,\ \sigma_n\in\mathscr{B}_n} Q_\alpha(\rho_n\|\sigma_n)$$

$$C(\rho_n\|\sigma_n) := \max_{\alpha\in[0,1]} -\log Q_\alpha(\rho_n\|\sigma_n)$$

Extension 1

$$C(\mathscr{A}_n\|\ \mathscr{B}_n) := \inf_{\rho_n\in\mathscr{A}_n,\ \sigma_n\in\mathscr{B}_n} C(\rho_n\|\sigma_n)$$

Extension 2

$$\mathfrak{C}_{n,r}(\mathscr{A}_n\|\mathscr{B}_n) := \max_{\alpha\in[0,1]} -\log Q_\alpha(\mathscr{A}_n\|\mathscr{B}_n)$$

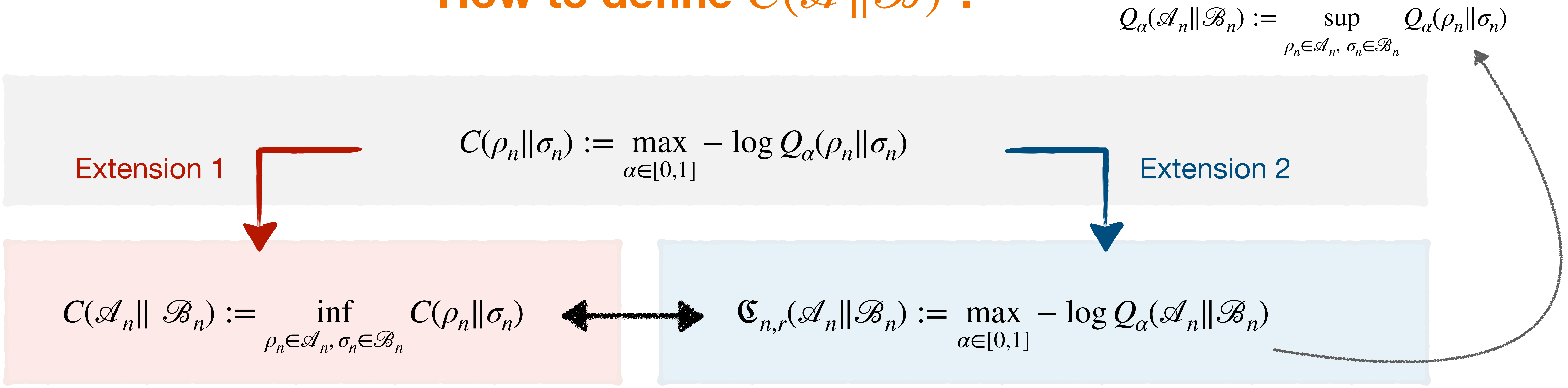
Are they equivalent?

$$\inf_{\rho_n\in\mathscr{A}_n,\ \sigma_n\in\mathscr{B}_n} \max_{\alpha\in[0,1]} -\log Q_\alpha(\rho_n\|\sigma_n) \underset{\text{minimax theorem}}{=\!=} \max_{\alpha\in[0,1]} \inf_{\rho_n\in\mathscr{A}_n,\ \sigma_n\in\mathscr{B}_n} -\log Q_\alpha(\rho_n\|\sigma_n)$$

Concave in (ρ_n, σ_n), convex in α

(IV) Chernoff exponent (2508.12889)

How to define $C(\mathscr{A}\|\mathscr{B})$?



$$Q_\alpha(\mathscr{A}_n\|\mathscr{B}_n) := \sup_{\rho_n\in\mathscr{A}_n,\ \sigma_n\in\mathscr{B}_n} Q_\alpha(\rho_n\|\sigma_n)$$

$$C(\rho_n\|\sigma_n) := \max_{\alpha\in[0,1]} -\log Q_\alpha(\rho_n\|\sigma_n)$$

Extension 1

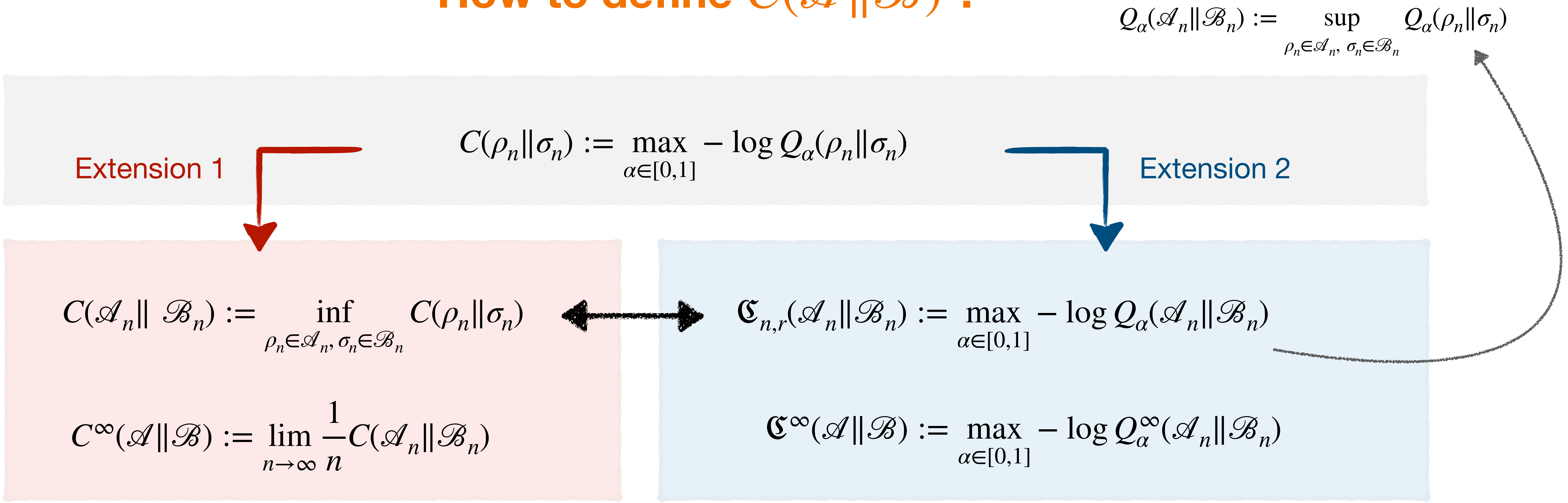
Extension 2

$$C(\mathscr{A}_n\|\ \mathscr{B}_n) := \inf_{\rho_n\in\mathscr{A}_n,\, \sigma_n\in\mathscr{B}_n} C(\rho_n\|\sigma_n)$$

$$\mathfrak{C}_{n,r}(\mathscr{A}_n\|\mathscr{B}_n) := \max_{\alpha\in[0,1]} -\log Q_\alpha(\mathscr{A}_n\|\mathscr{B}_n)$$

(IV) Chernoff exponent (2508.12889)

How to define $C(\mathscr{A}\|\mathscr{B})$?



$$Q_\alpha(\mathscr{A}_n\|\mathscr{B}_n) := \sup_{\rho_n\in\mathscr{A}_n,\ \sigma_n\in\mathscr{B}_n} Q_\alpha(\rho_n\|\sigma_n)$$

$$C(\rho_n\|\sigma_n) := \max_{\alpha\in[0,1]} -\log Q_\alpha(\rho_n\|\sigma_n)$$

Extension 1

Extension 2

$$C(\mathscr{A}_n\|\ \mathscr{B}_n) := \inf_{\rho_n\in\mathscr{A}_n,\, \sigma_n\in\mathscr{B}_n} C(\rho_n\|\sigma_n)$$

$$C^\infty(\mathscr{A}\|\mathscr{B}) := \lim_{n\to\infty}\frac{1}{n}C(\mathscr{A}_n\|\mathscr{B}_n)$$

$$\mathfrak{C}_{n,r}(\mathscr{A}_n\|\mathscr{B}_n) := \max_{\alpha\in[0,1]} -\log Q_\alpha(\mathscr{A}_n\|\mathscr{B}_n)$$

$$\mathfrak{C}^\infty(\mathscr{A}\|\mathscr{B}) := \max_{\alpha\in[0,1]} -\log Q_\alpha^\infty(\mathscr{A}_n\|\mathscr{B}_n)$$

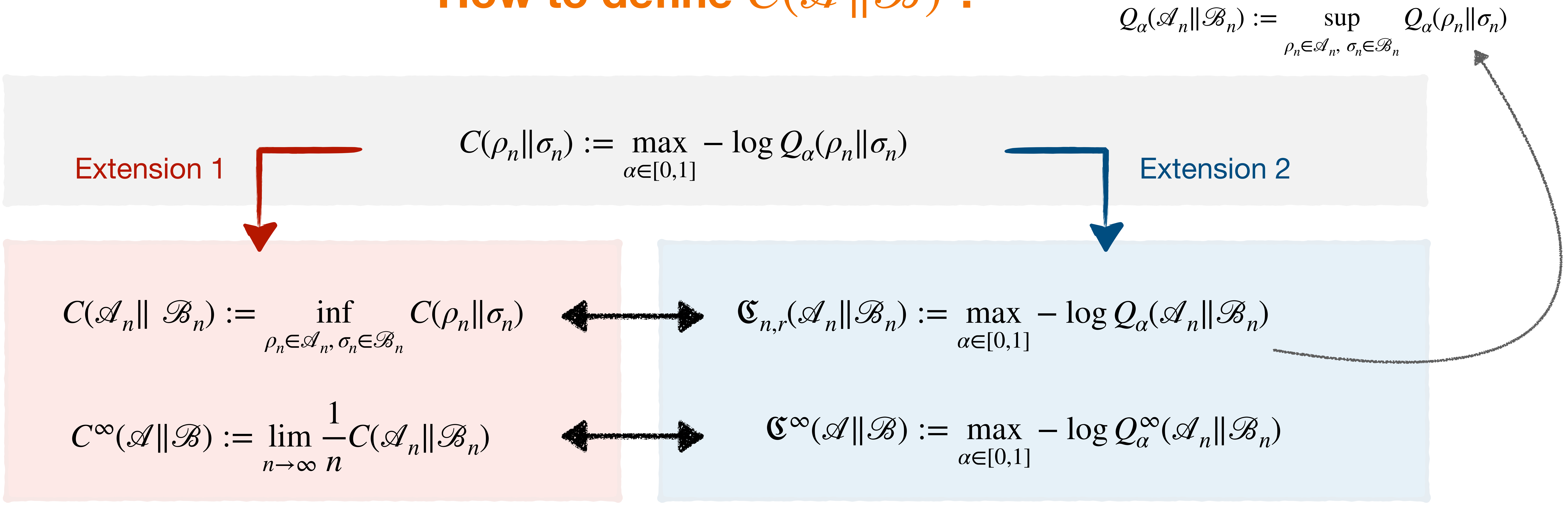
Are they equivalent?

$$\inf_{n\geq 1}\ \max_{\alpha\in[0,1]}\ \underbrace{-\frac{1}{n}\log Q_\alpha(\mathscr{A}_n\|\mathscr{B}_n)}\ \underset{\text{minimax theorem [Mosonyi-Hiai-11]}}{=}\ \max_{\alpha\in[0,1]}\ \inf_{n\geq 1} -\frac{1}{n}\log Q_\alpha(\mathscr{A}_n\|\mathscr{B}_n)$$

Upper semicontinuous in α, monotone decreasing in k, $n = 2^k$

(IV) Chernoff exponent (2508.12889)

How to define $C(\mathscr{A}\|\mathscr{B})$?



$$Q_\alpha(\mathscr{A}_n\|\mathscr{B}_n) := \sup_{\rho_n\in\mathscr{A}_n,\ \sigma_n\in\mathscr{B}_n} Q_\alpha(\rho_n\|\sigma_n)$$

$$C(\rho_n\|\sigma_n) := \max_{\alpha\in[0,1]} -\log Q_\alpha(\rho_n\|\sigma_n)$$

Extension 1

Extension 2

$$C(\mathscr{A}_n\|\ \mathscr{B}_n) := \inf_{\rho_n\in\mathscr{A}_n,\, \sigma_n\in\mathscr{B}_n} C(\rho_n\|\sigma_n) \longleftrightarrow \mathfrak{C}_{n,r}(\mathscr{A}_n\|\mathscr{B}_n) := \max_{\alpha\in[0,1]} -\log Q_\alpha(\mathscr{A}_n\|\mathscr{B}_n)$$

$$C^\infty(\mathscr{A}\|\mathscr{B}) := \lim_{n\to\infty} \frac{1}{n} C(\mathscr{A}_n\|\mathscr{B}_n) \longleftrightarrow \mathfrak{C}^\infty(\mathscr{A}\|\mathscr{B}) := \max_{\alpha\in[0,1]} -\log Q_\alpha^\infty(\mathscr{A}_n\|\mathscr{B}_n)$$

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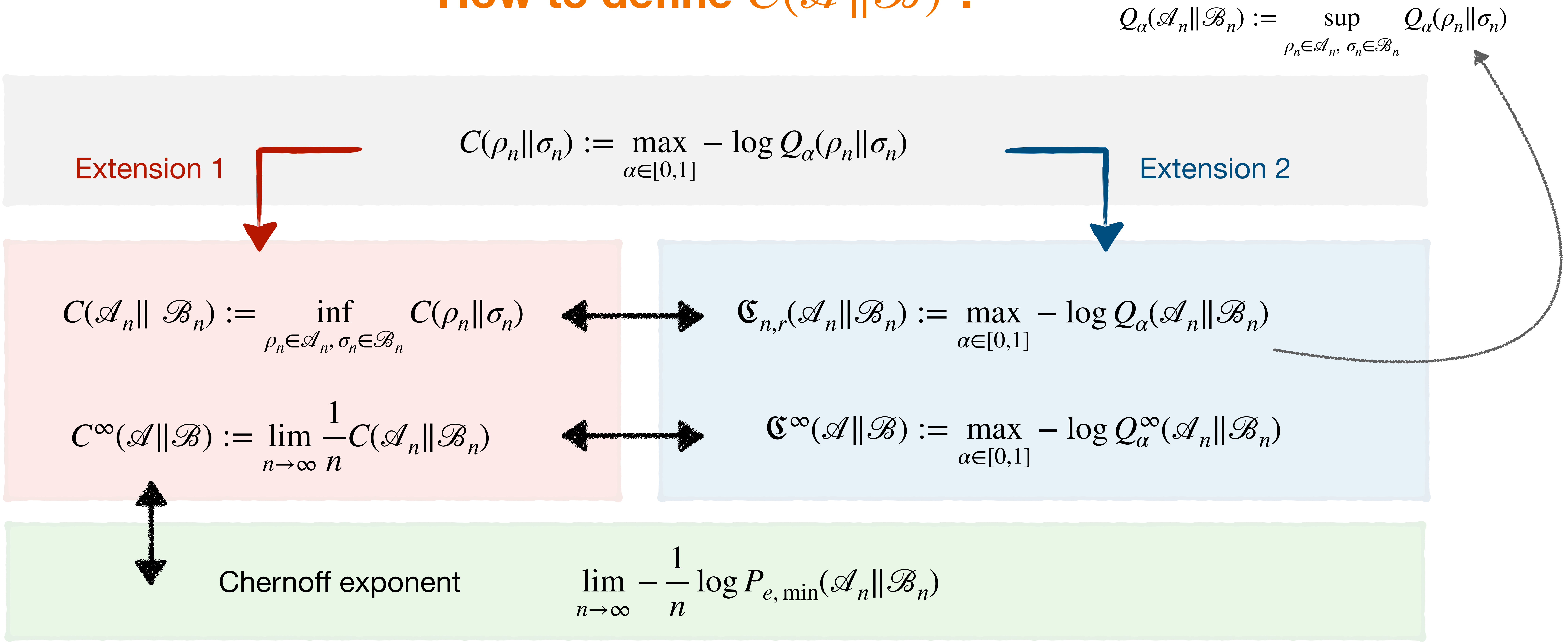

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Chernoff exponent

$$\lim_{n\to\infty} -\frac{1}{n}\log P_{e,\min}(\mathscr{A}_n\|\mathscr{B}_n)$$

Same result holds for **multiple hypotheses.**

This extends the result by [Li-16]. Refer to the full paper on arXiv: 2508.12889.

Summary

★ (A.1) Each $\mathscr{B}_n$ is convex and compact;

(A.3) $\mathscr{B}_m \otimes \mathscr{B}_k \subseteq \mathscr{B}_{m+k}$, for all $m, k \in \mathbb{N}$;

Operational regimes	**Information measure**	**IID Sources**	**Composite Correlated**
(I) Stein's exponent	Quantum relative entropy	[Hiai-Petz-91] [Ogawa-Nagaoka-00]	[Hayashi-Yamasaki-24] [Lami-24] [KF-Fawzi-Fawzi-24]
(II) Error exponent	Quantum Hoeffding divergence	[Ogawa-Hayashi-02] [Hayashi-06][Nagaoka-06] [Audenaert et al.-07]	**2508.12901 (this talk)**
(III) Strong converse exponent	Quantum Hoeffding anti-divergence	[Ogawa-Nagaoka-00] [Mosonyi-Ogawa-15]	**2508.12901 (this talk)**
(IV) Chernoff exponent	Quantum Chernoff divergence	[Audenaert et al.-07] [Nussbaum-Szkoła-09]	**2508.12889 (this talk)**

An almost complete picture for composite correlated hypotheses

(Extremely general; recover the i.i.d. setting as a special case)

Missing part: (III) achievable part of strong converse exponent for general set $\mathscr{A}_n$

Applications: explore the black-box/adversarial/correlated settings for any tasks utilizing the quantum hypothesis testing e.g. channel coding, channel discrimination, data compression, resource theory …

Entanglement theory: 2601.10190; Thermodynamics: 2604.13524; 2606.06318 (@Kaito Watanabe's talk on Tuesday)

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Thanks for your attention!

arXiv:2508.12901 & 2508.12889