On some extensions of the generalised quantum Stein's lemma

L. / Regula / Takagi, 2605.15174

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European Research Council
Established by the European Commission

Introduction to quantum hypothesis testing

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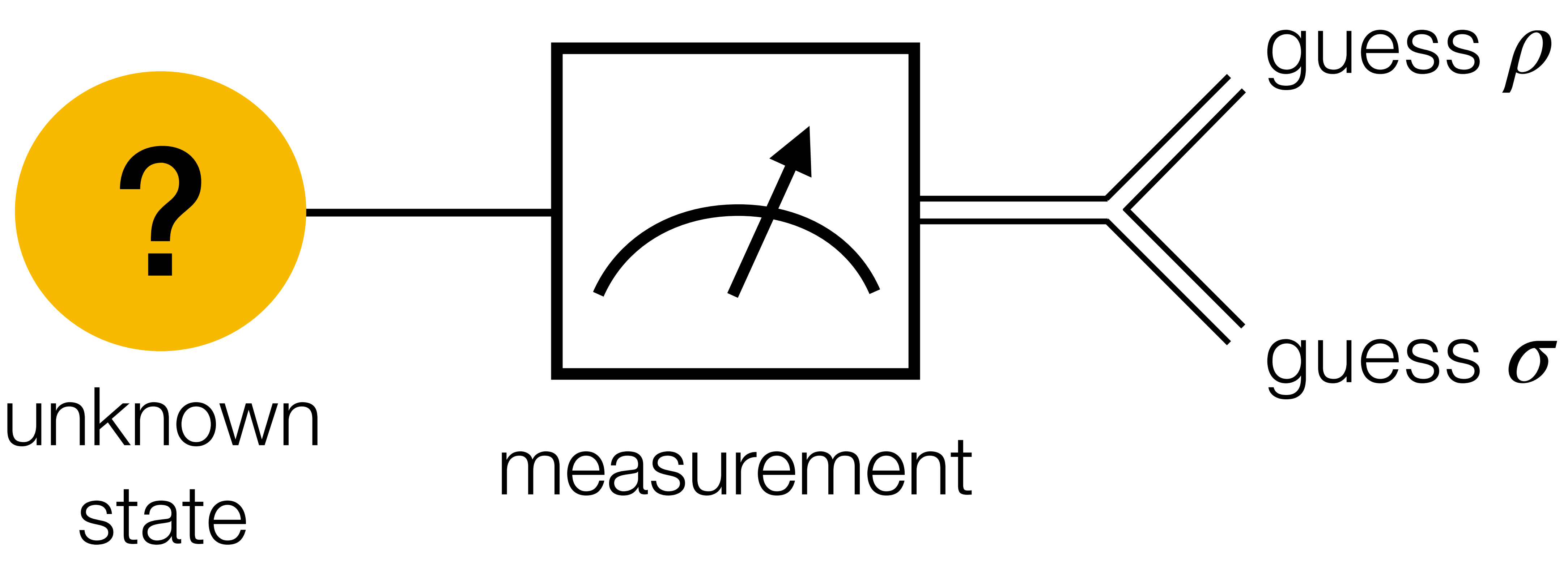
Alternative hypothesis: σ



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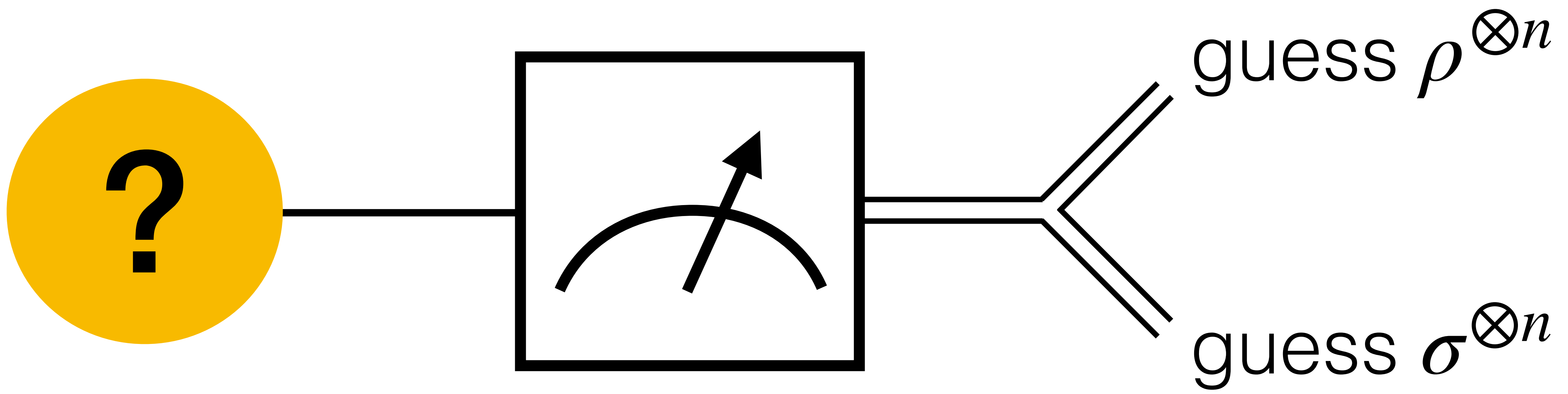
Simple IID hypotheses: many copies of one of two states.

Null hypothesis: $\rho^{\otimes n}$

Alternative hypothesis: $\sigma^{\otimes n}$



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Composite hypotheses: two **sets** of quantum states.

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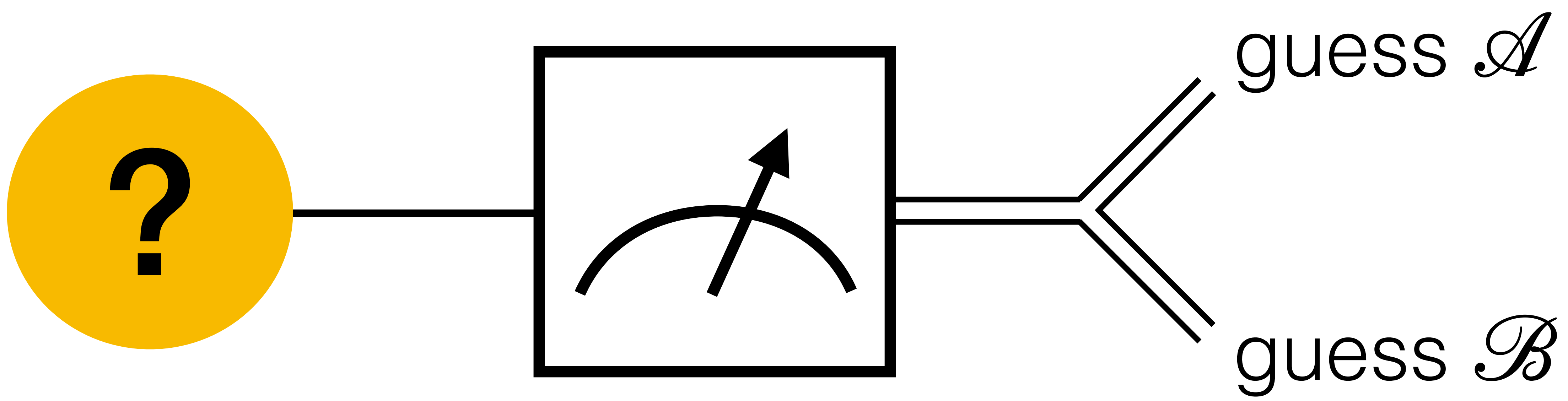

$\mathscr{A}, \mathscr{B}$: sets of states on the same quantum system.

Quantum hypothesis testing = Given two hypotheses concerning a quantum state, which one holds true?

Composite hypotheses: two sequences of sets of states, indexed by $n \in \mathbb{N}$.

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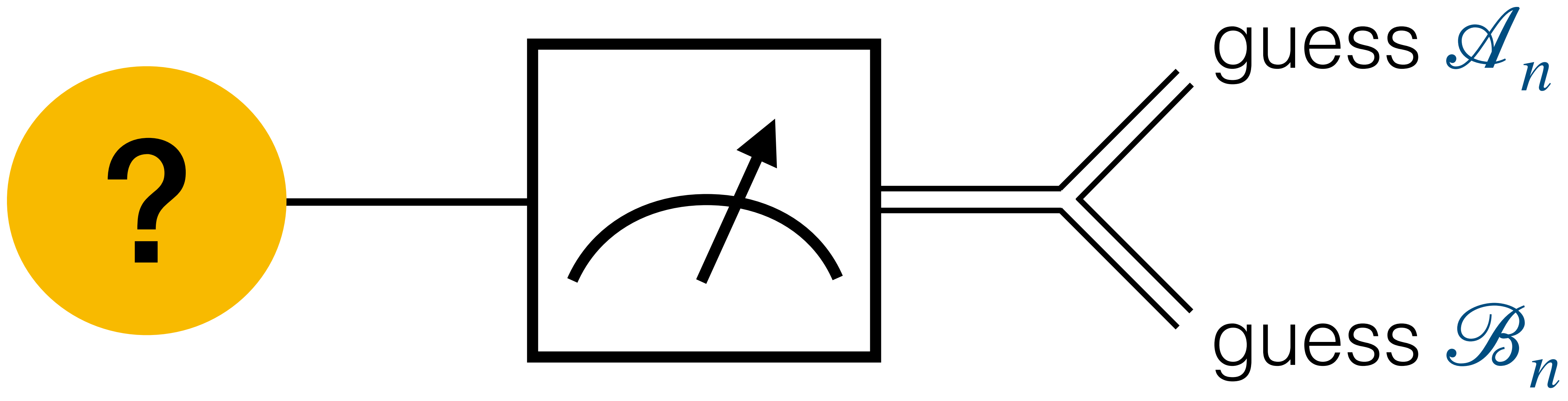

$\mathscr{A}_n, \mathscr{B}_n \subseteq \mathscr{D}\left(\mathscr{H}^{\otimes n}\right)$ sequences of sets of states on the n-copy system.

Two types of error:

- Type 1: null hypothesis correct, guessed alternative
- Type 2: alternative hypothesis correct, guessed null

Null hypothesis: $\rho_n \in \mathscr{A}_n$

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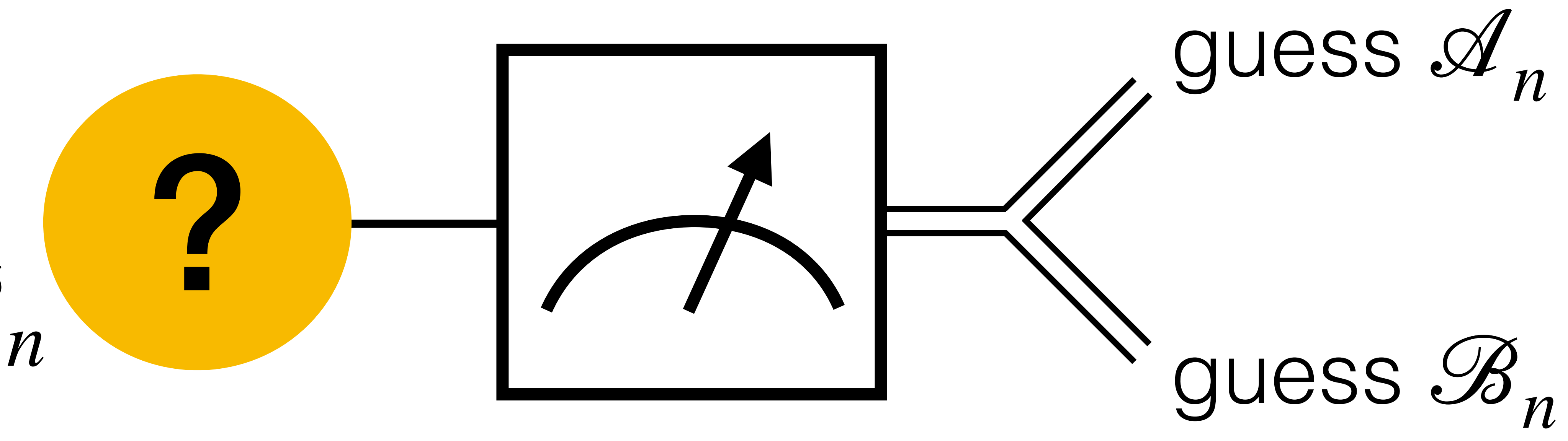

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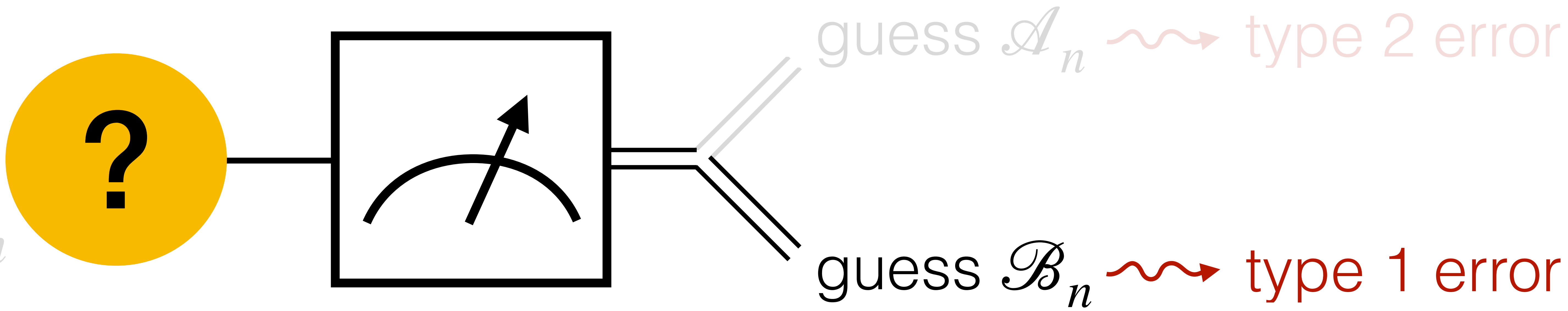

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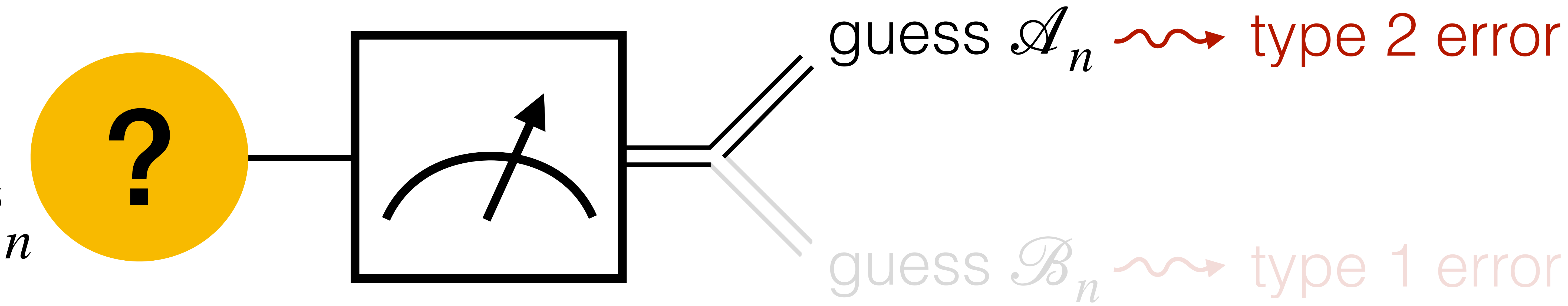

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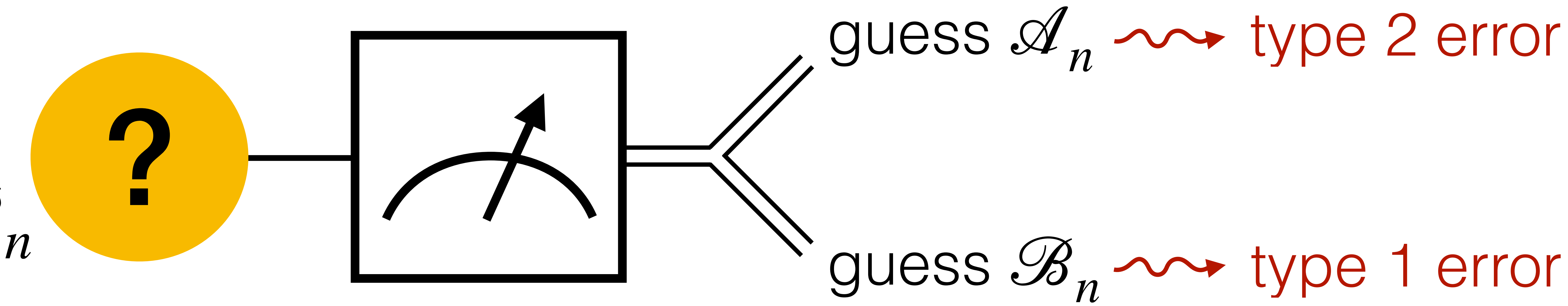

Asymmetric hypothesis testing: fix Pr{type 1} $\leq \varepsilon$,
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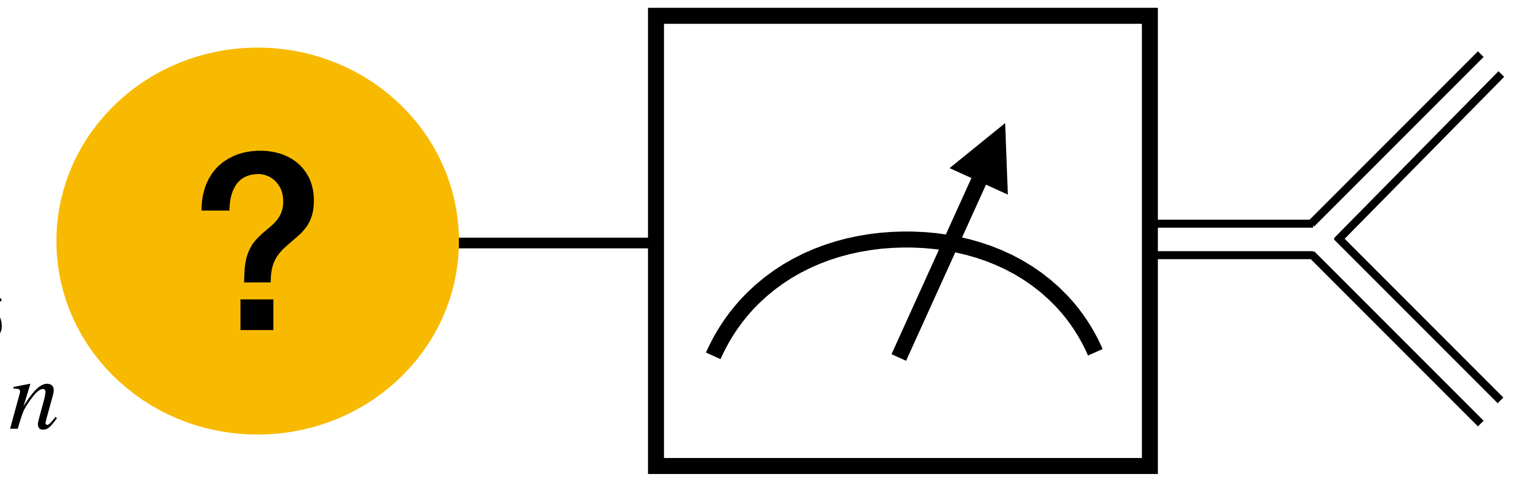
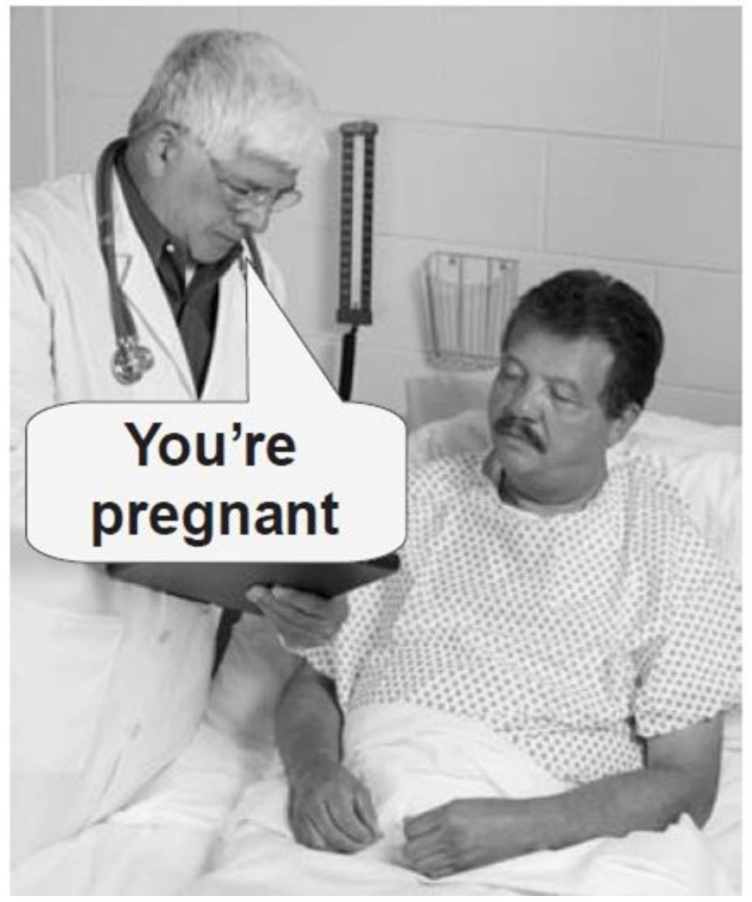


guess $\mathscr{A}_n \rightsquigarrow$ type 2 error

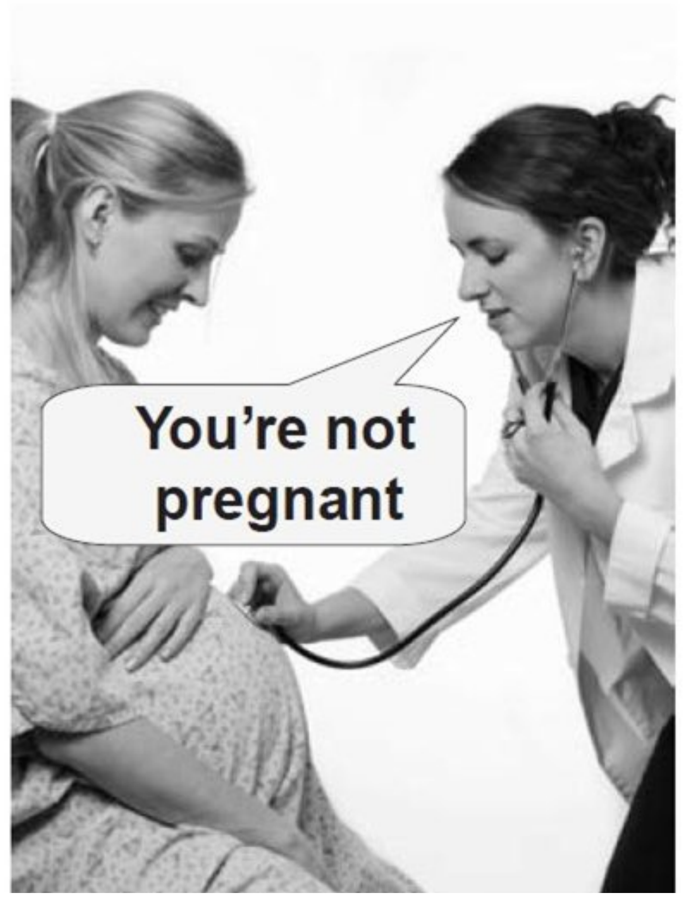
guess $\mathscr{B}_n \rightsquigarrow$ type 1 error

Asymmetric hypothesis testing: fix Pr{type 1} $\leq \varepsilon$, minimise Pr{type 2} as $n \to \infty$.

Type I error
(false positive)



Type II error
(false negative)



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Stein exponent:

$$\mathrm{Stein}(\mathscr{A}\|\mathscr{B}) := \lim_{\varepsilon\to 0}\lim_{n\to\infty} -\frac{1}{n}\log\min\left\{\Pr\{\text{type 2}\} : \ \Pr\{\text{type 1}\} \leq \varepsilon\right\}$$

$$\mathscr{A} = (\mathscr{A}_n)_n\,, \qquad \mathscr{B} = (\mathscr{B}_n)_n$$

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Problem: compute $\mathrm{Stein}(\mathscr{A}\|\mathscr{B})$ for the most general sequences of sets $\mathscr{A}, \mathscr{B}$.

(**Compute :=** find an "entropic" formula = eliminate the limit $\varepsilon \to 0$.)

Typically, such formulas involve the **quantum relative entropy**:

$$D(\rho\|\sigma) := \mathrm{Tr}\left[\rho(\log\rho - \log\sigma)\right]$$

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- Genuinely correlated sets: extremal points are not product across the copies.

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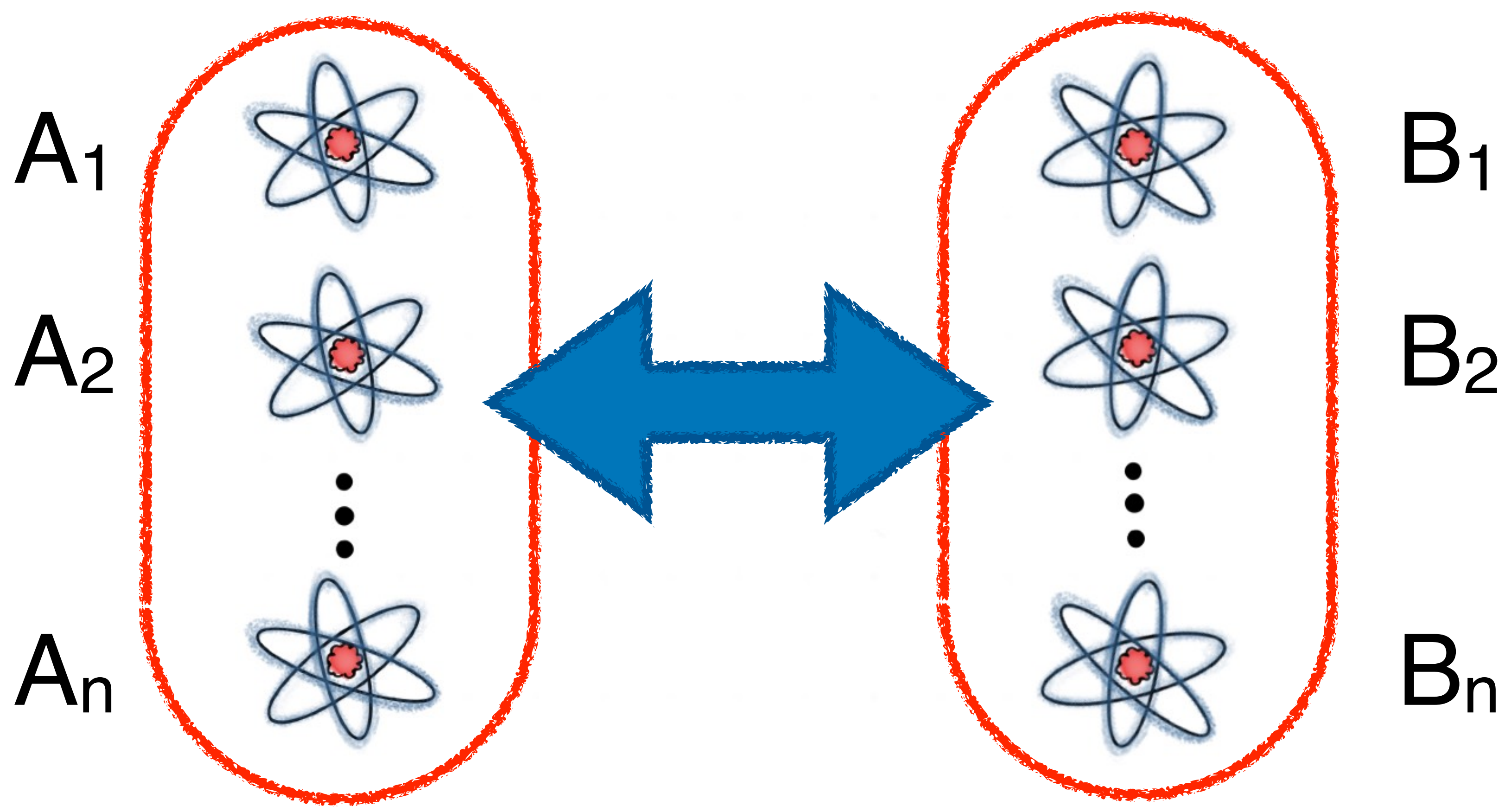
Example of more complicated sets:

Separable states (SEP)



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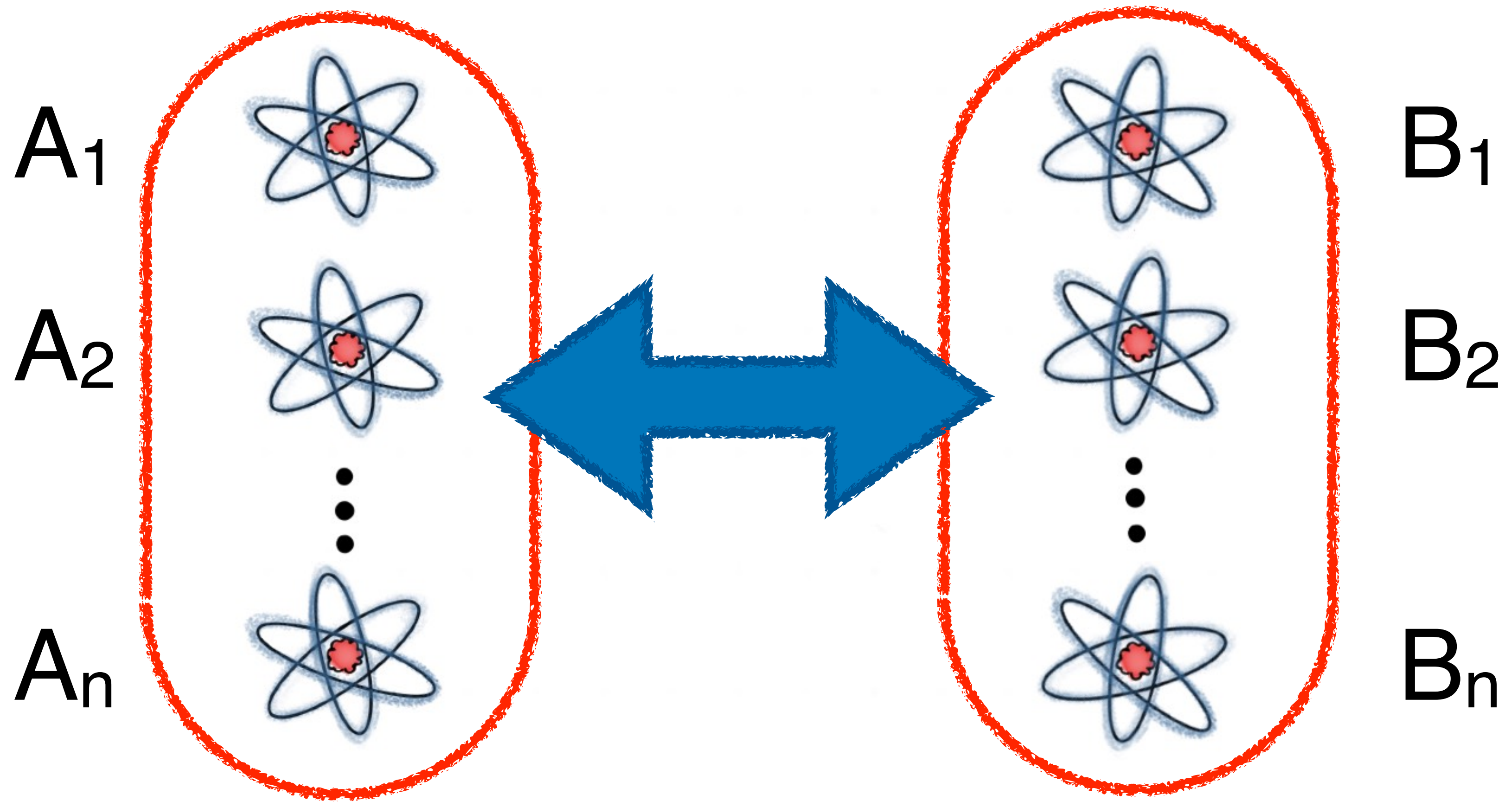

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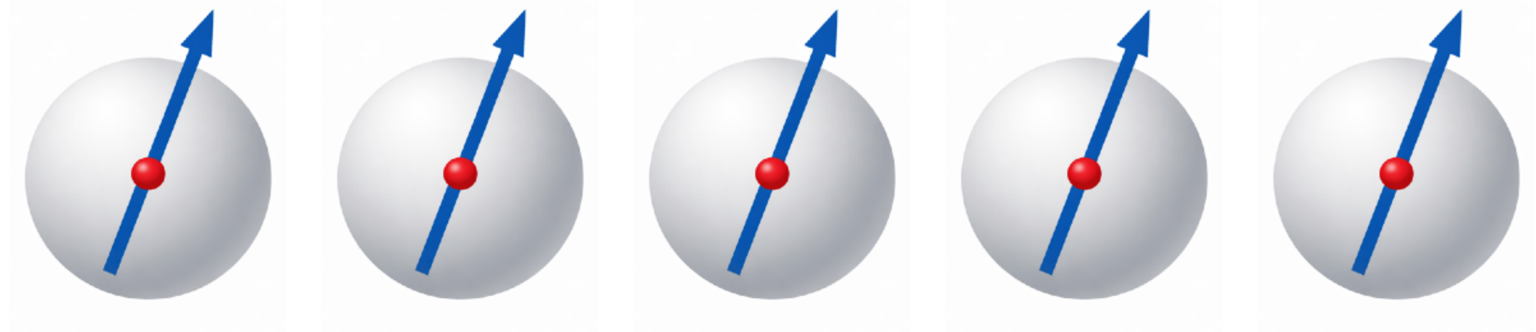
Stabiliser states (STAB)

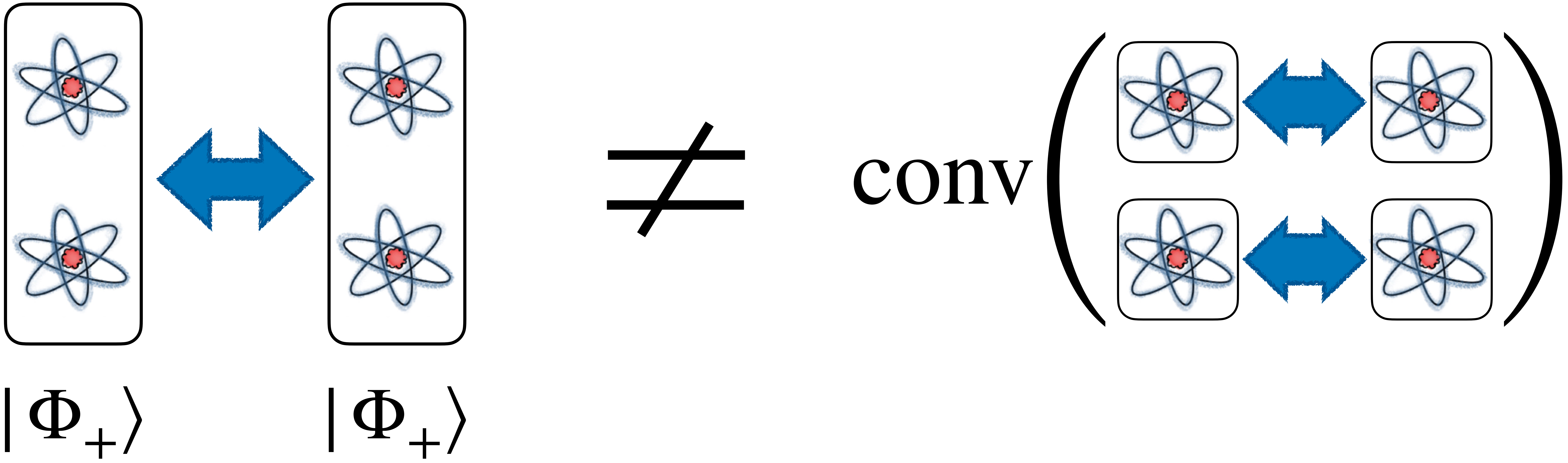


n-qubit system

Clifford group

$$\mathrm{STAB}_n := \mathrm{conv}\left\{ C|0\rangle\langle 0|^{\otimes n} C^\dagger : \ C \in \mathscr{C}_n \right\}$$

Genuinely correlated: extremal points are **not** product across the copies.



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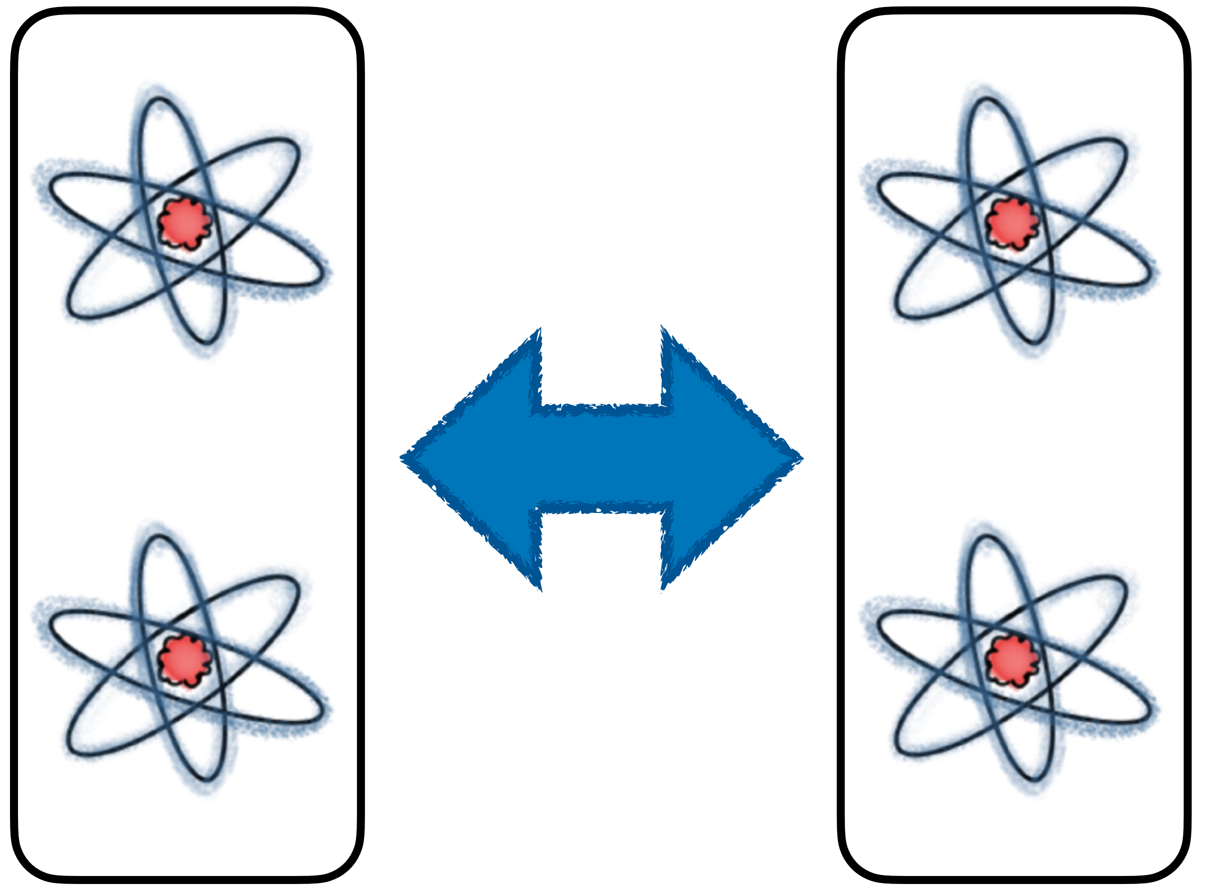
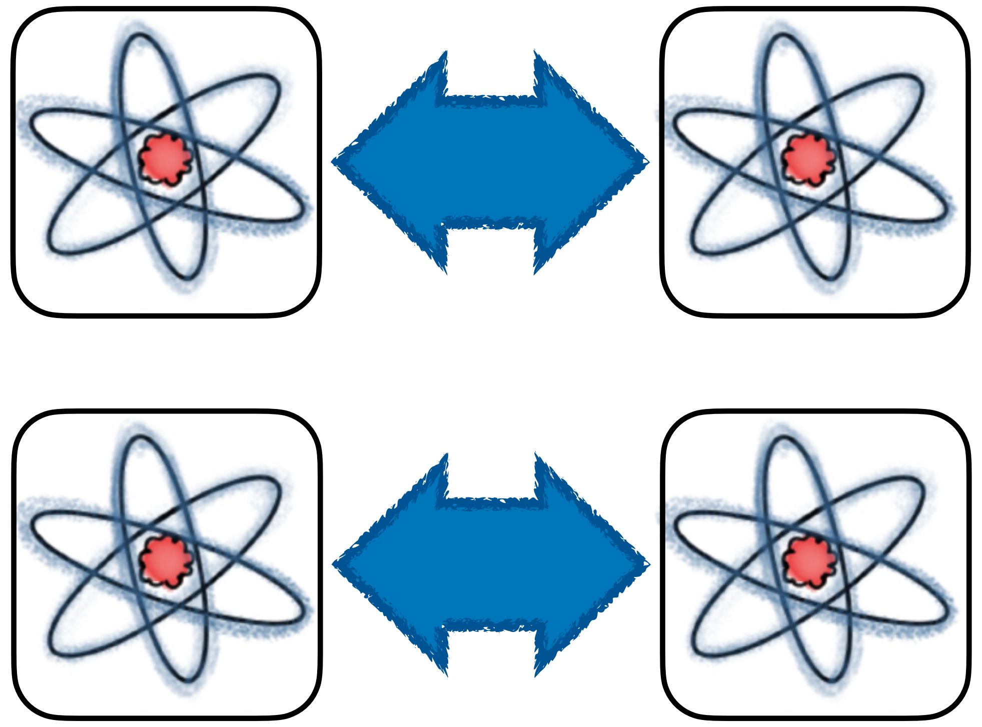


$|\Phi_+\rangle$ $\quad$ $|\Phi_+\rangle$

$\neq \mathrm{conv}\Big($



$\Big)$

$$\sum_i p_i \, \sigma^{(i)}_{A_1:B_1} \otimes \sigma^{(i)}_{A_2:B_2}, \quad \sigma^{(i)}_{A_j:B_j} \in \mathrm{SEP}_1$$

Some prior results:

- Quantum Stein's lemma → simple IID vs simple IID:

$$\mathrm{Stein}\left(\rho^{\mathrm{iid}} \,\|\, \sigma^{\mathrm{iid}}\right) = D(\rho\|\sigma)$$

[Hiai/Petz 1991]

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- Quantum Sanov theorem → composite IID vs simple IID:

$$\mathrm{Stein}\left(\mathscr{A}^{\mathrm{iid}} \,\|\, \rho^{\mathrm{iid}}\right) = D(\mathscr{A} \,\|\, \rho) = \inf_{\sigma\in\mathscr{A}} D(\sigma\|\rho)$$

[Bjelakovič et al. 2005]

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Note: this is just the minimum Stein exponent of the simple IID problems!

⟶ Composite IID-ness of the null hypothesis simplifies away!

- Generalised quantum Stein's lemma → simple IID vs SEP/STAB

$$\mathrm{Stein}\left(\rho^{\mathrm{iid}} \,\|\, \mathrm{SEP}\right) = D^{\infty}(\rho\|\mathrm{SEP}) := \lim_{n\to\infty} \frac{1}{n} \inf_{\sigma_n \in \mathrm{SEP}_n} D\left(\rho^{\otimes n} \,\|\, \sigma_n\right)$$

[Brandão/Plenio 2010]
[Hayashi/Yamasaki 2025]
[L. 2025]

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Correlations in the alternative hypothesis

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- Generalised quantum Sanov theorem → SEP/STAB vs simple IID:

$$\mathrm{Stein}\big(\mathrm{SEP} \,\|\, \rho^{\mathrm{iid}}\big) = D(\mathrm{SEP} \,\|\, \rho)$$

[L./Berta/Regula 2026]

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See also [Kanazawa/Yamasaki 2026]

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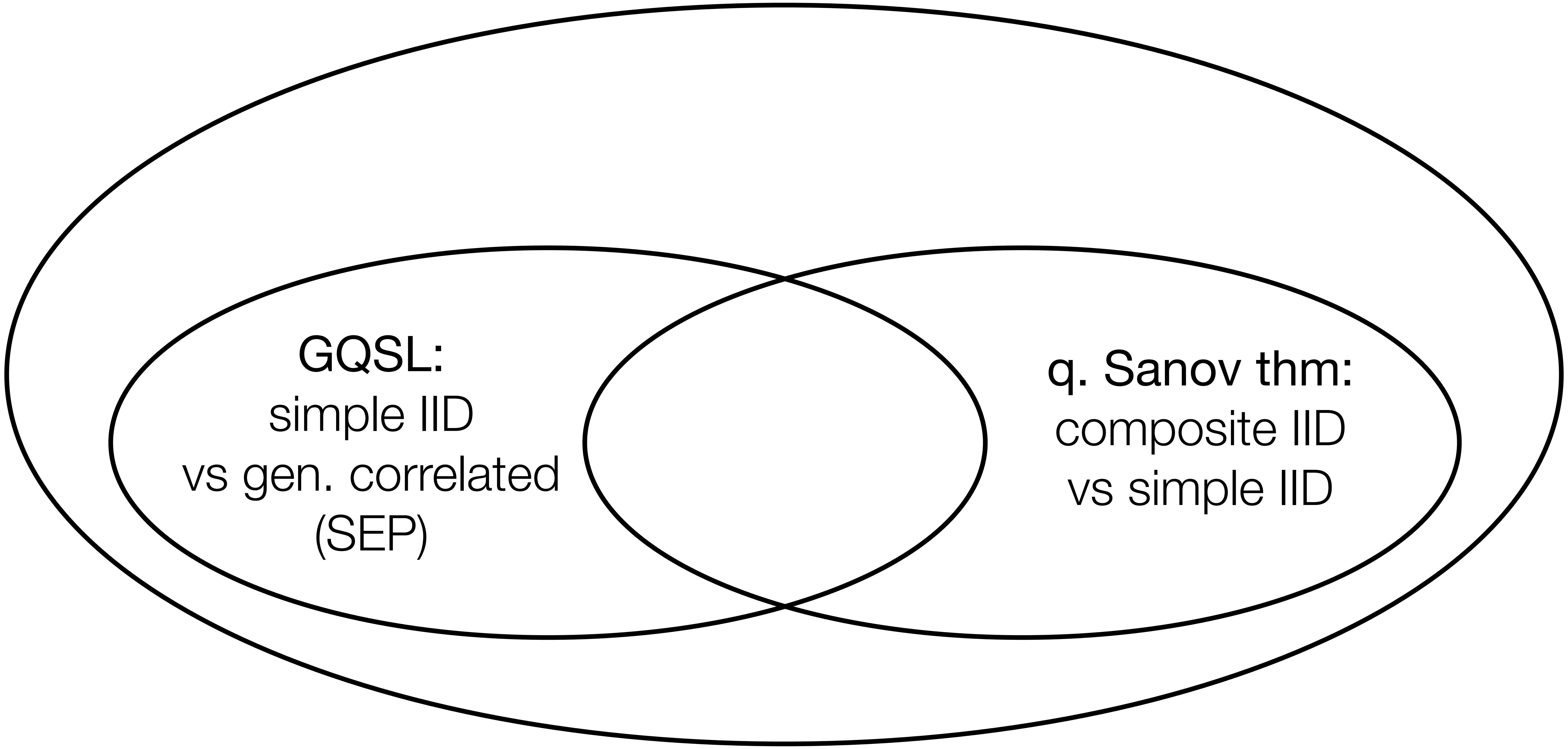
Does the problem reduce to the worst-case scenario over a fixed IID state $\rho \in \mathscr{A} \subseteq \mathscr{D}(\mathscr{H}_{AB})$ (as in the quantum Sanov theorem)?

$$\mathrm{Stein}\left(\mathscr{A}^{\mathrm{iid}} \,\|\, \mathrm{SEP}\right) \overset{?}{=} \inf_{\rho \in \mathscr{A}} D^{\infty}(\rho \| \mathrm{SEP})$$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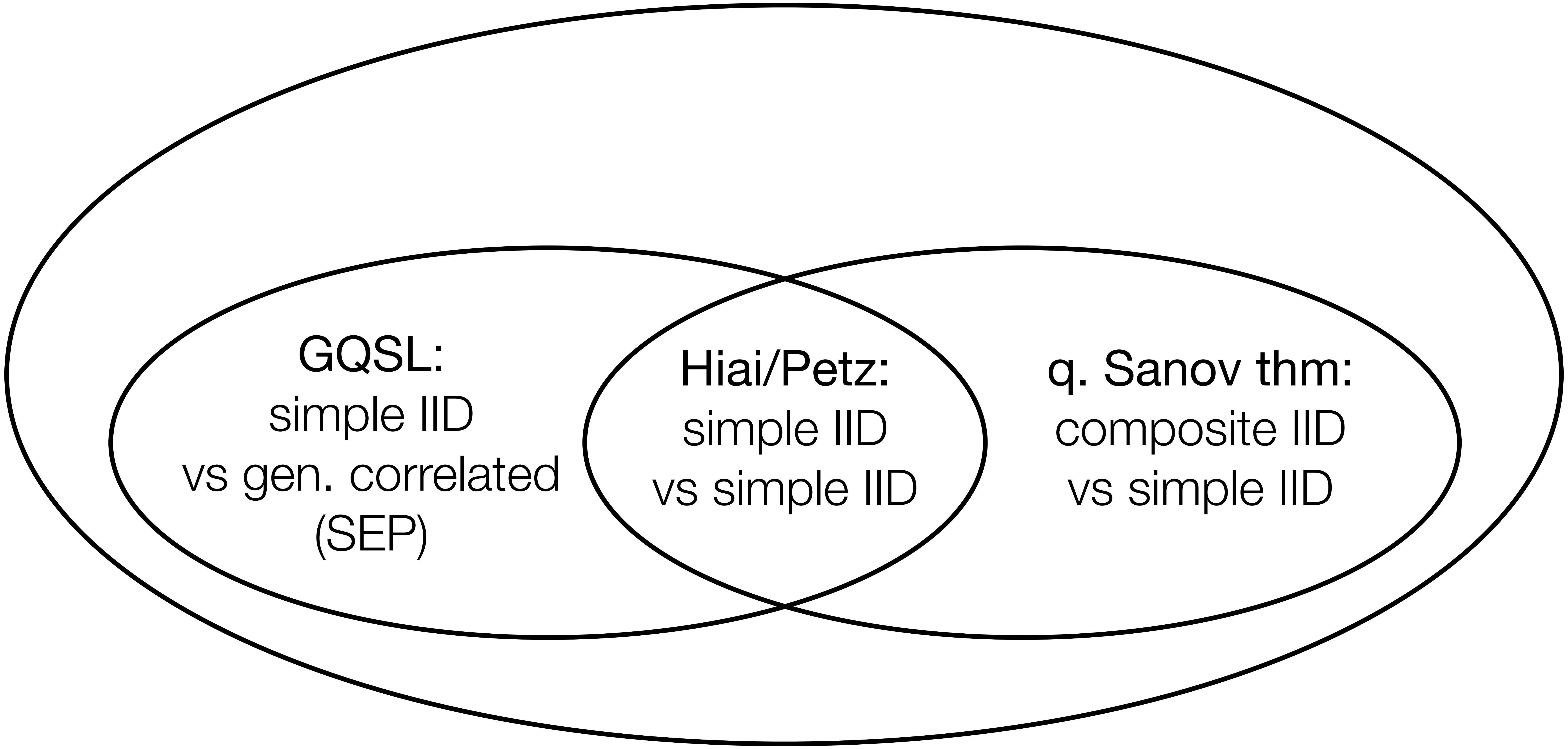
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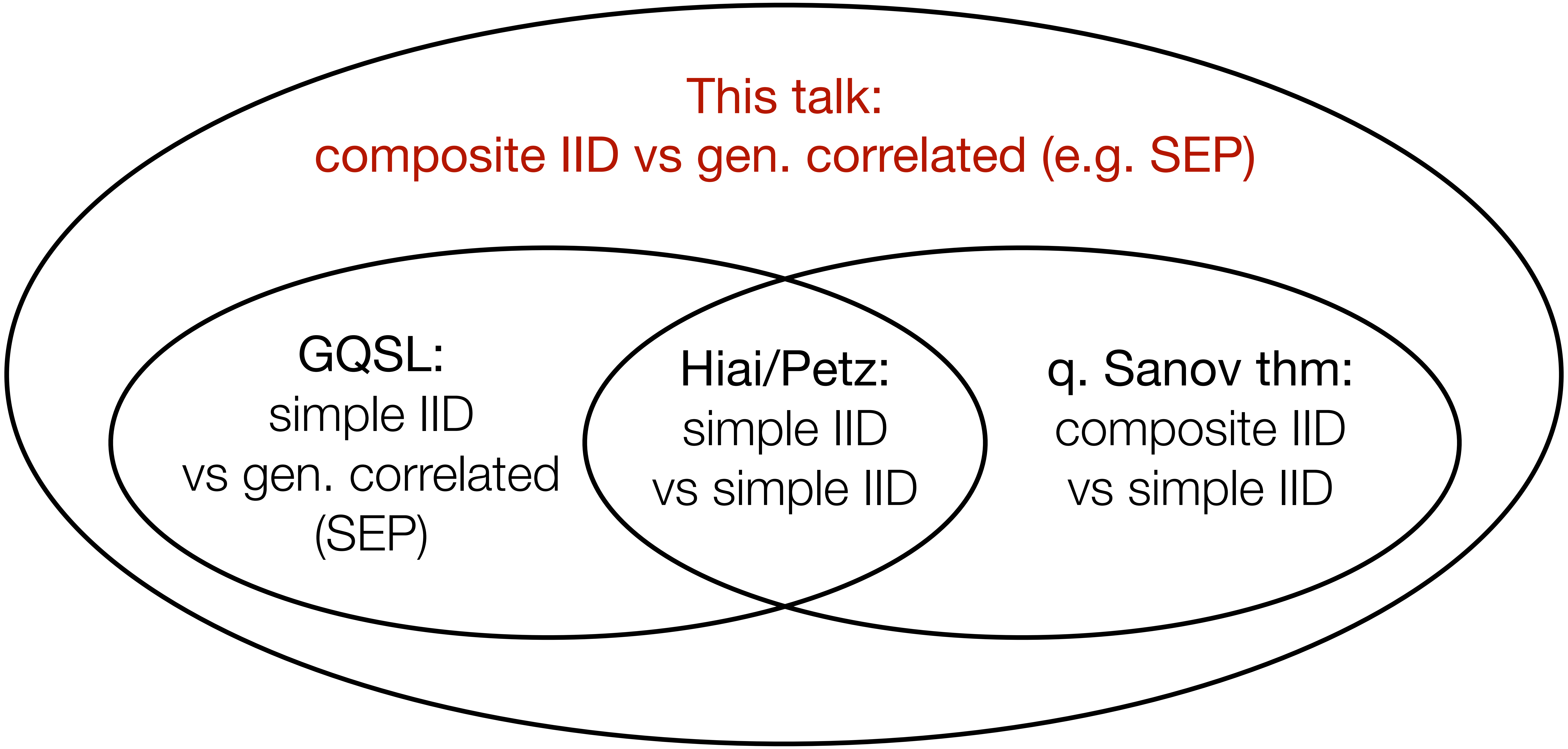
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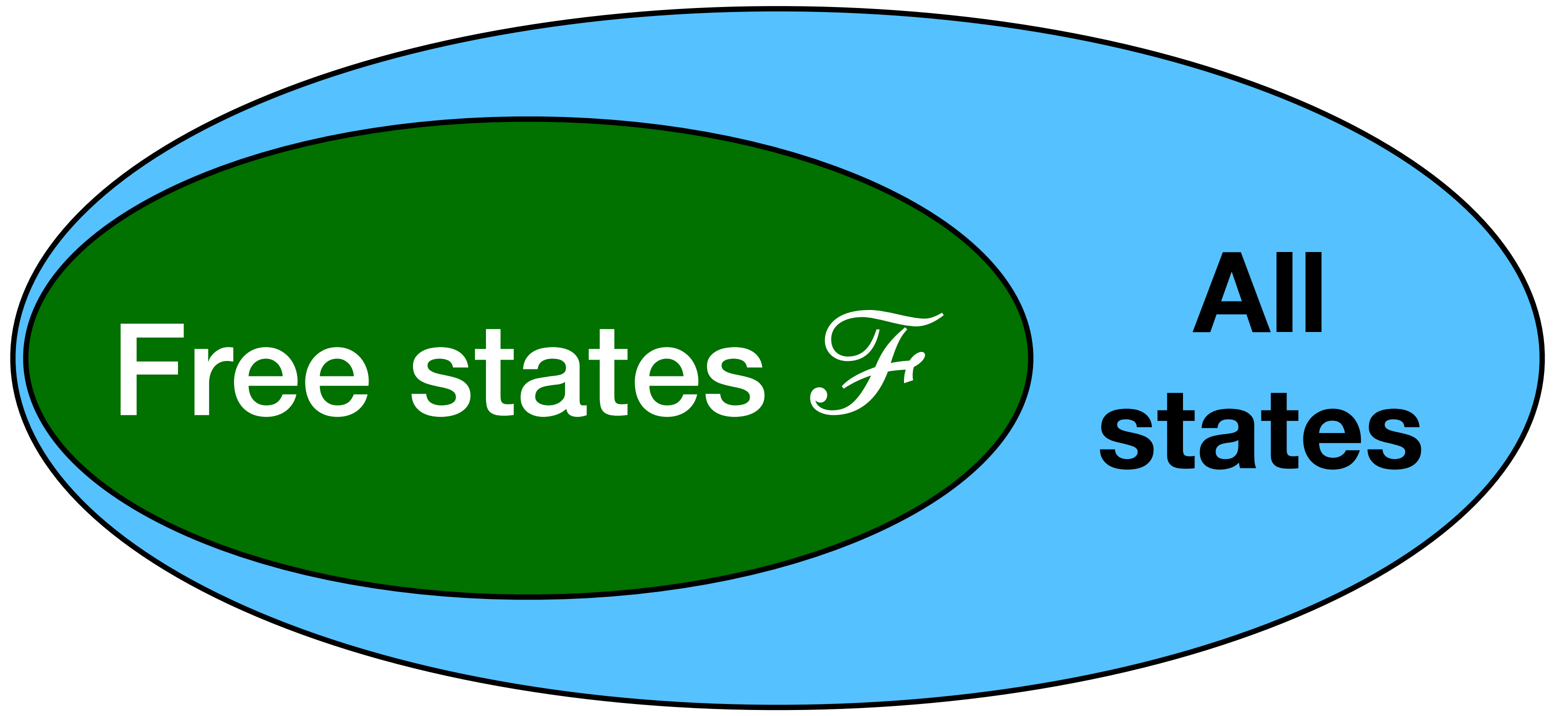
Motivation: universal resource manipulation

Quantum resource theories:

- A set of "free" (= easy to prepare) states.
- A set of "free operations" to manipulate states.



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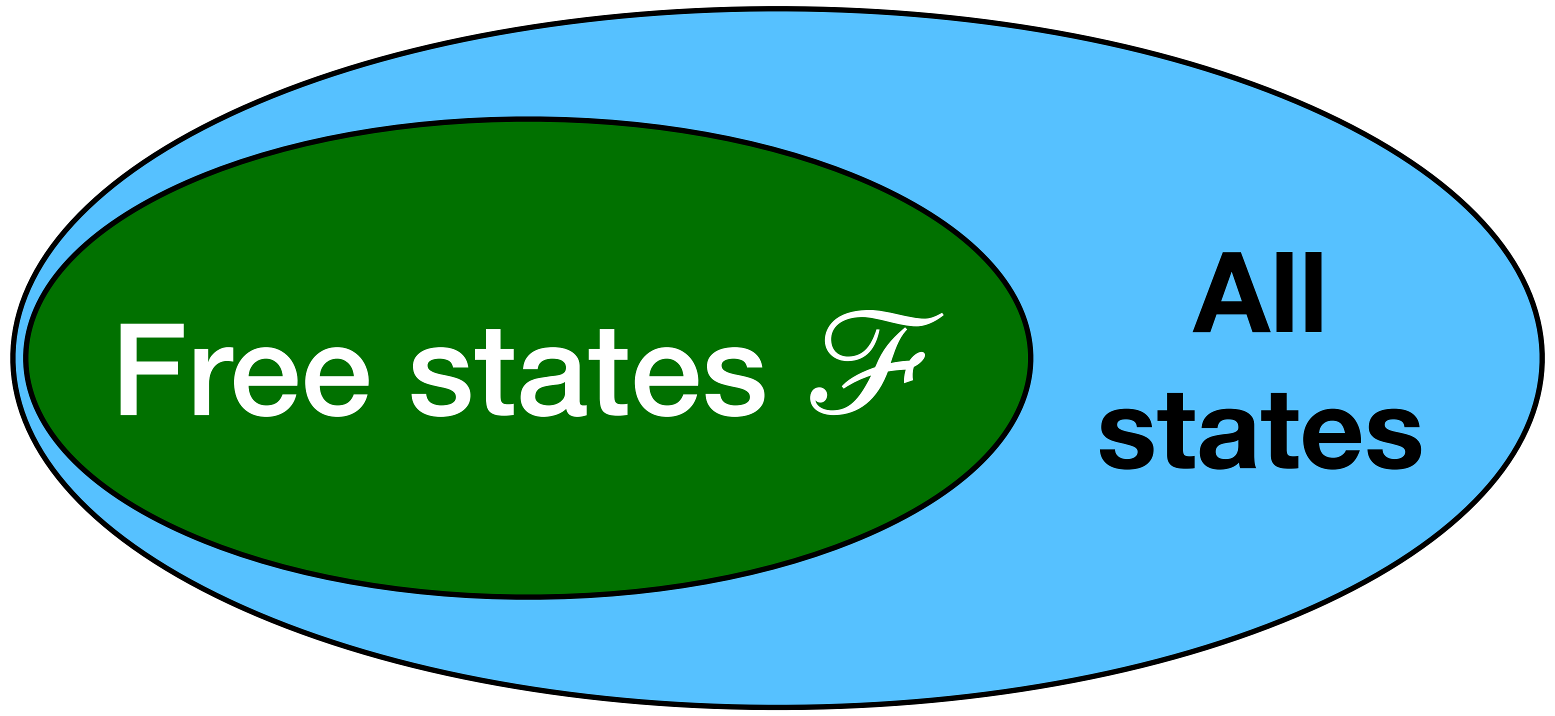

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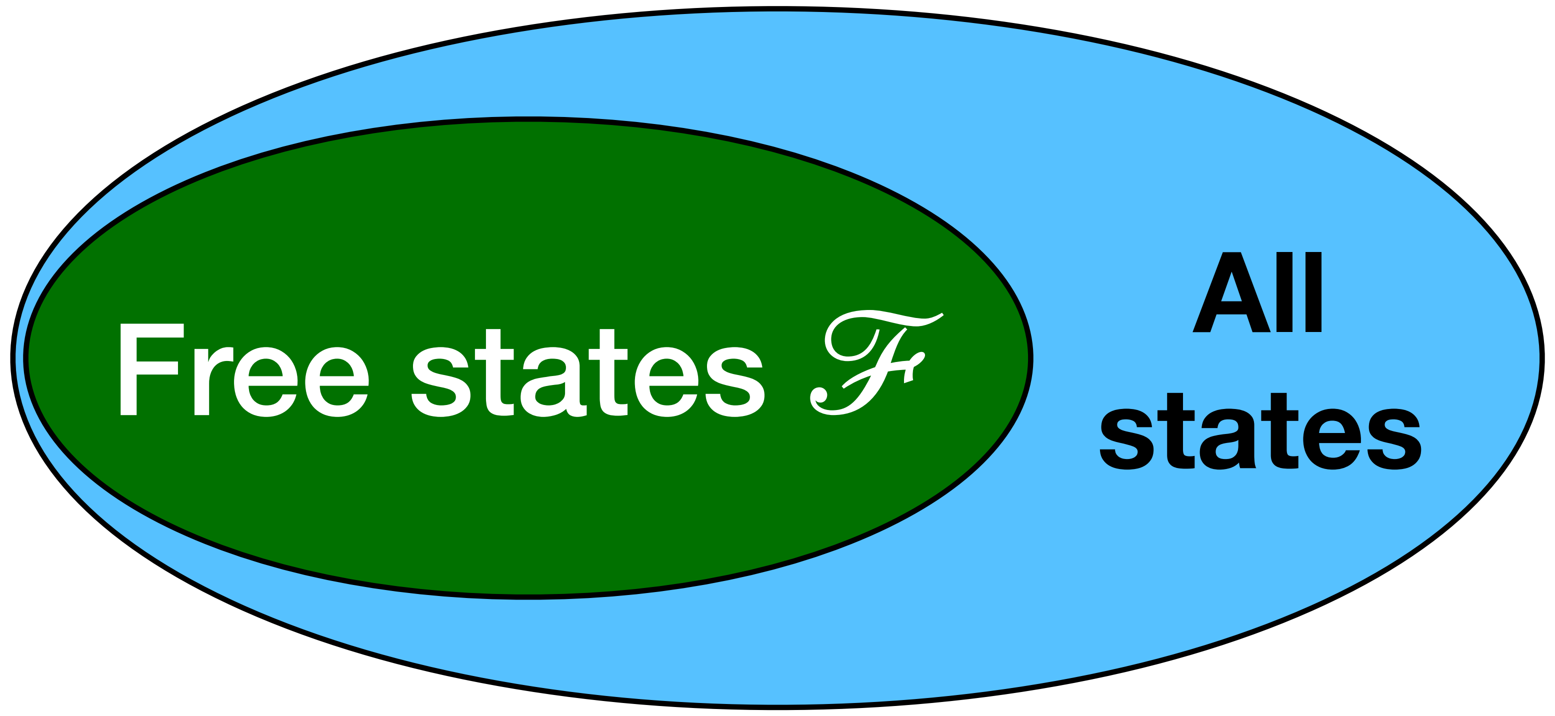

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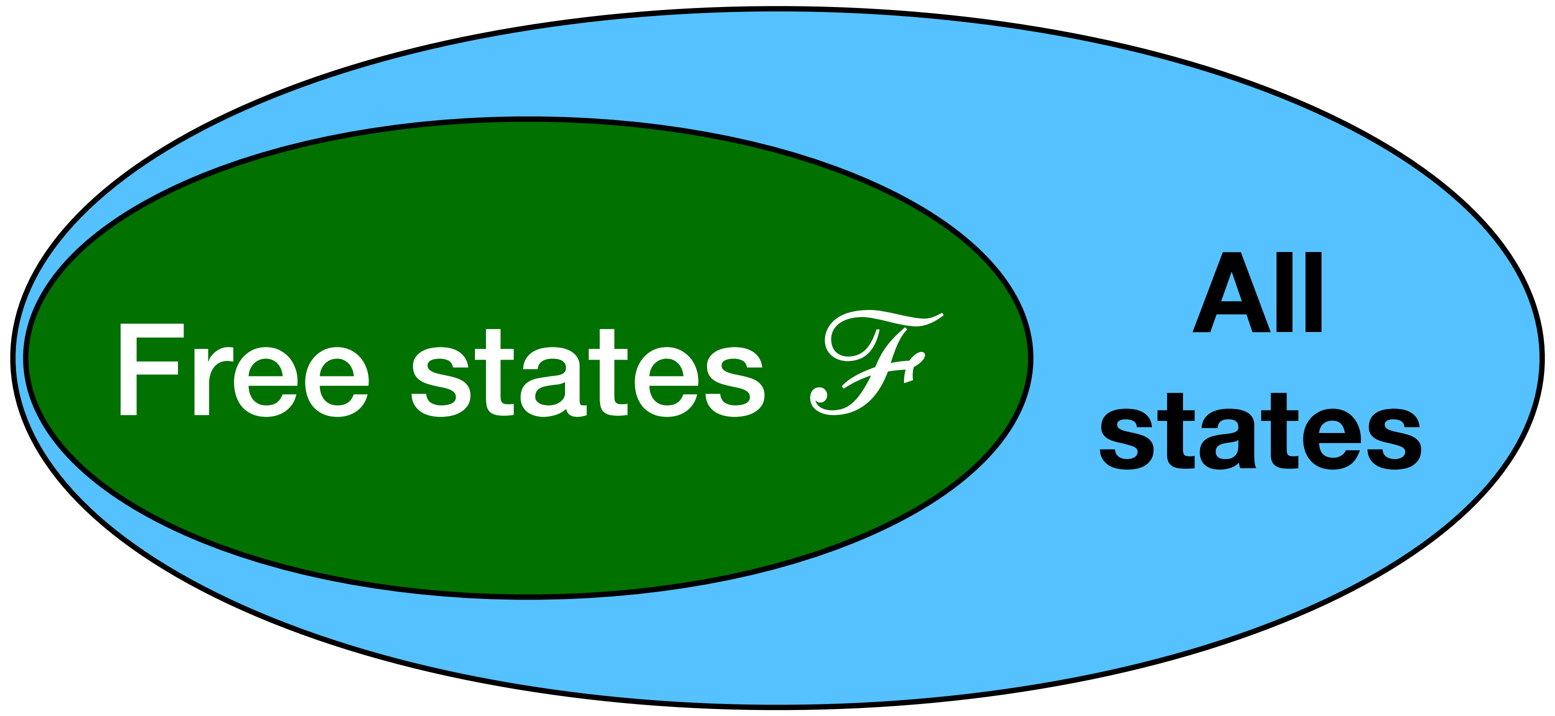

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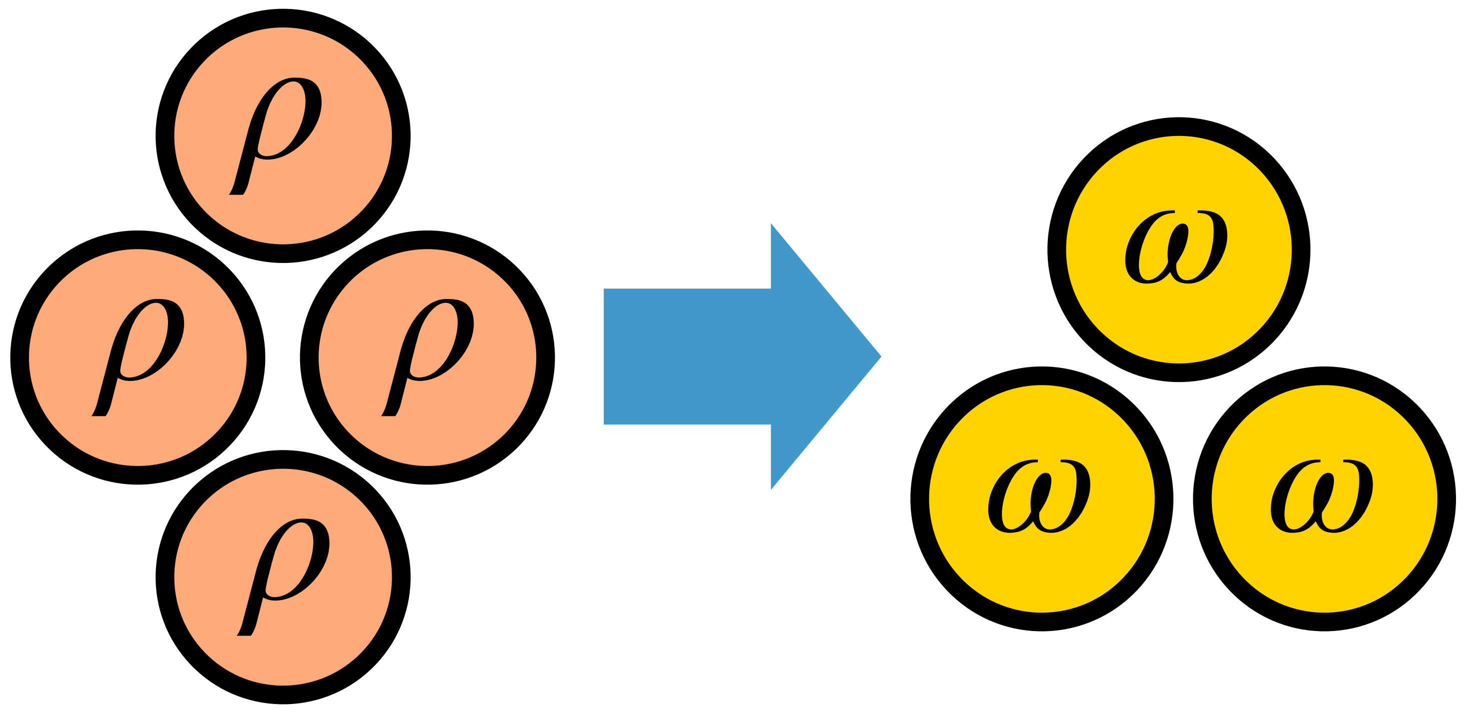
In practice, this is only imposed asymptotically (free operations can create a “vanishingly small” amount of resource): **ARNG operations**

[L./Regula 2023]

Quantum resource manipulation:

Transforming many copies of ρ into as many copies as possible of a target state ω using only free operations in $\mathcal{O}$.



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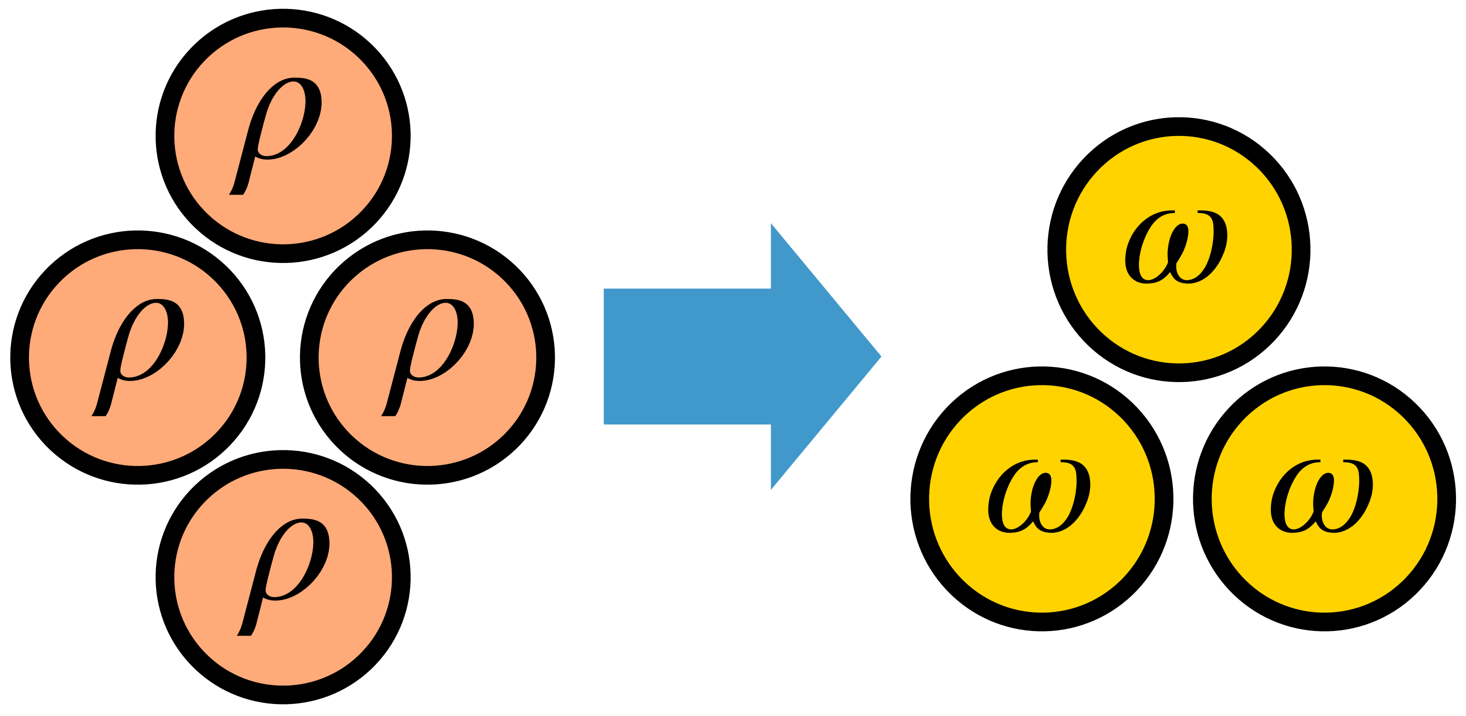

Maximum achievable rate (# copies ω per copy of ρ) w/ vanishing error:

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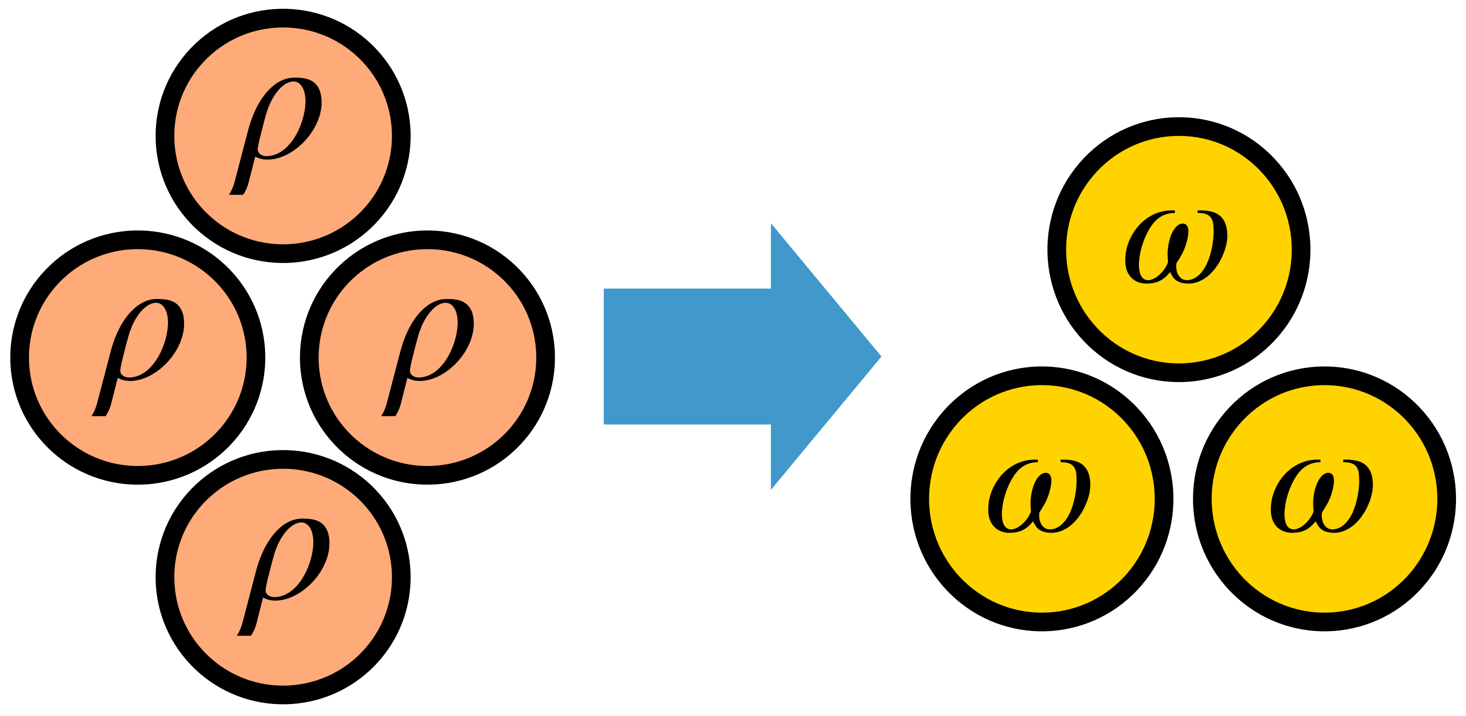

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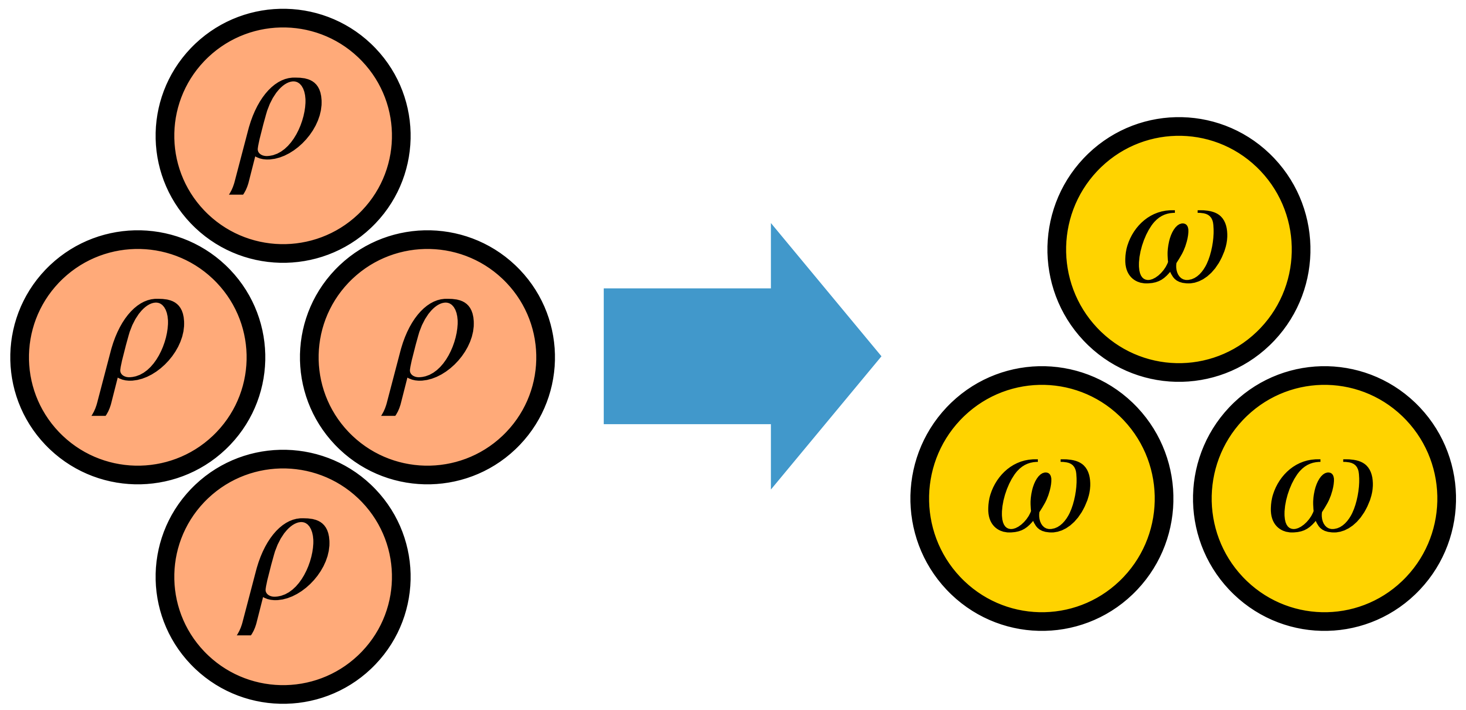
Brandão/Plenio: ARNG resource manipulation = resource testing

[Brandão/Plenio 2008 & 2010]

Quantum resource manipulation:

Transforming many copies of ρ into as many copies as possible of a target state ω using only free operations in $\mathcal{O}$.



Maximum achievable rate (# copies ω per copy of ρ) w/ vanishing error:

$$R_{\mathcal{O}}(\rho \to \omega)$$

Brandão/Plenio: ARNG resource manipulation = resource testing

$$R_{\text{ARNG}}(\rho \to \omega) = \frac{\text{Stein}(\rho\|\mathcal{F})}{D^{\infty}(\omega\|\mathcal{F})} \overset{\textbf{GQSL}}{=} \frac{D^{\infty}(\rho\|\mathcal{F})}{D^{\infty}(\omega\|\mathcal{F})}$$

[Brandão/Plenio 2008 & 2010]

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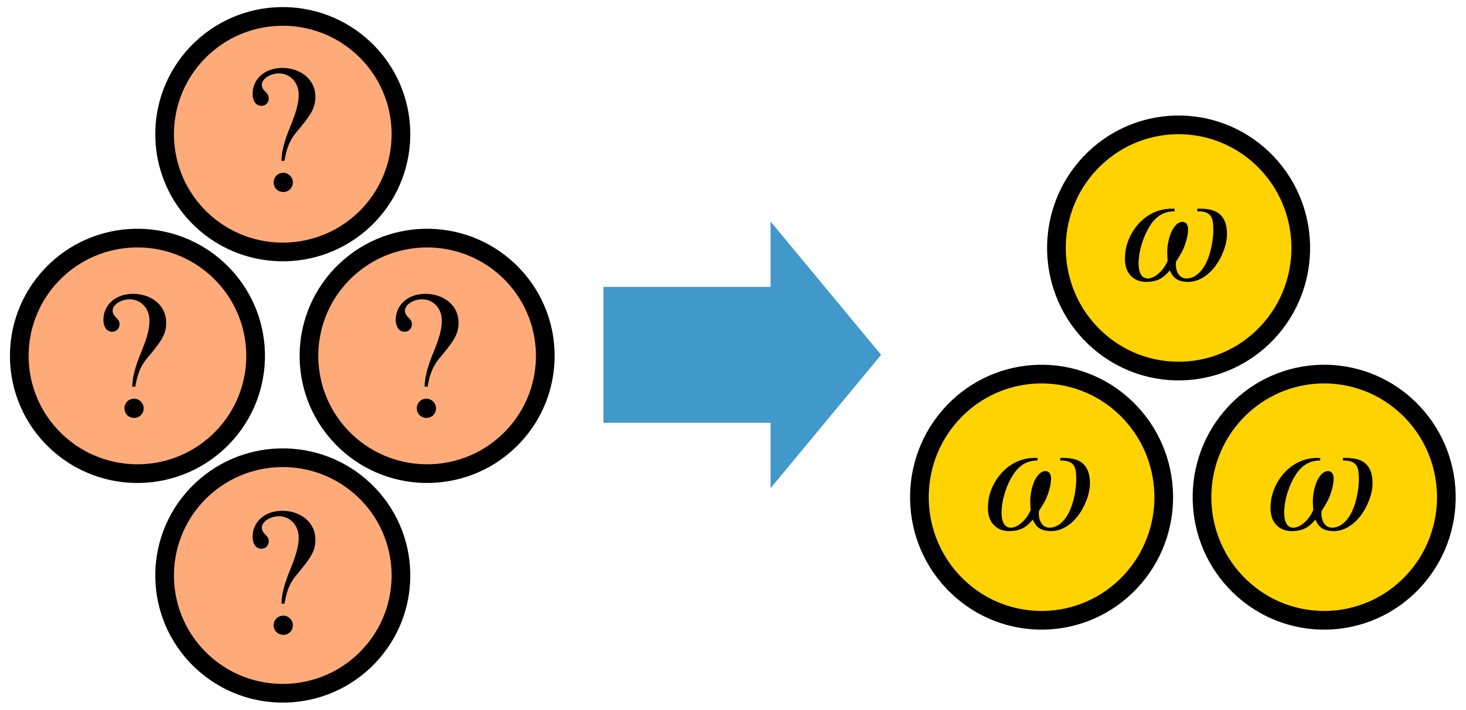

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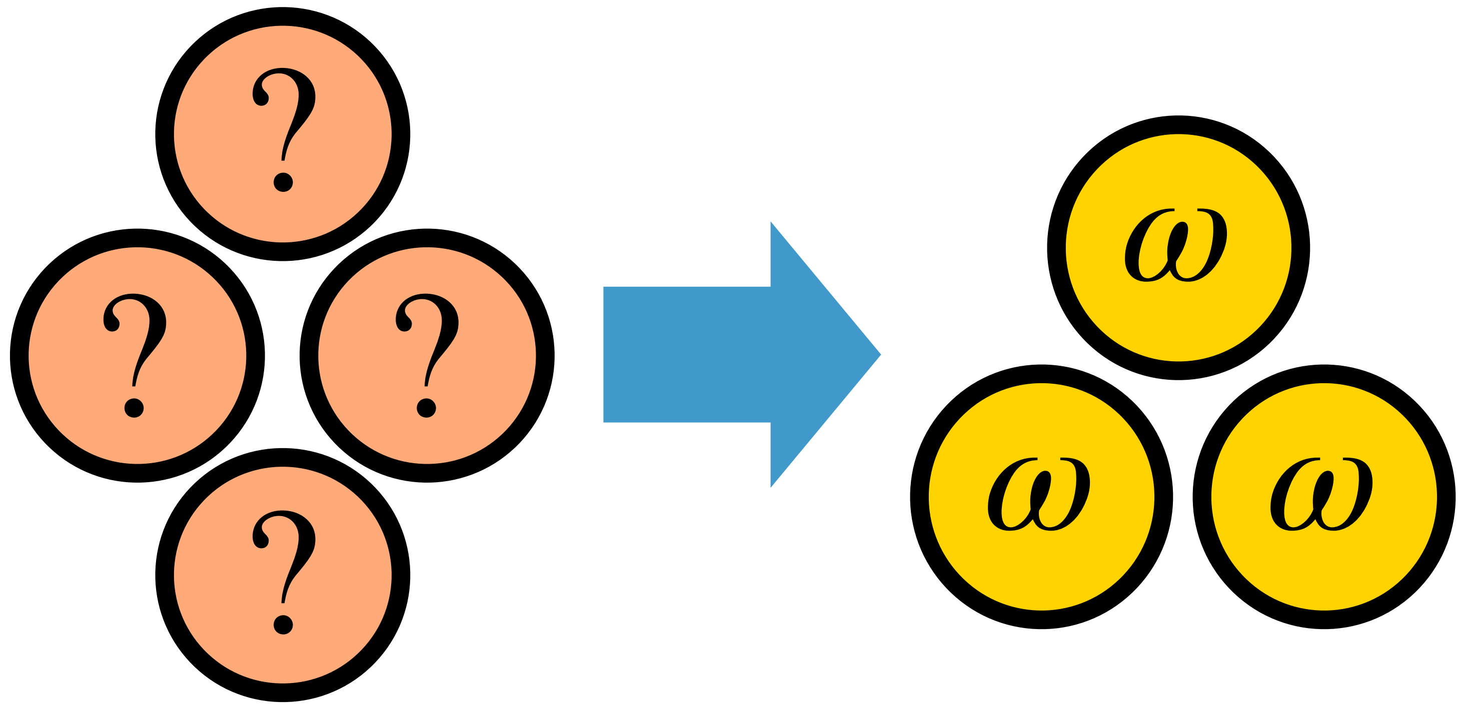

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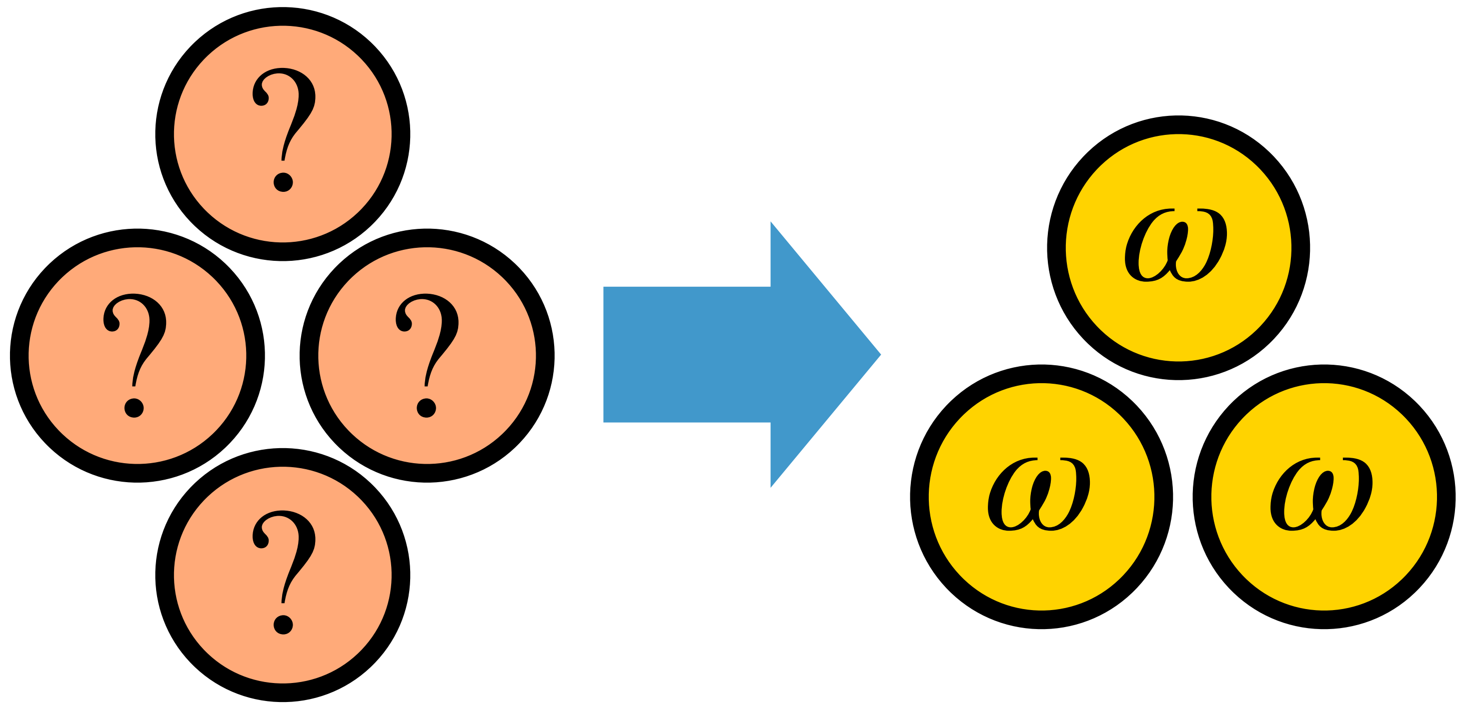

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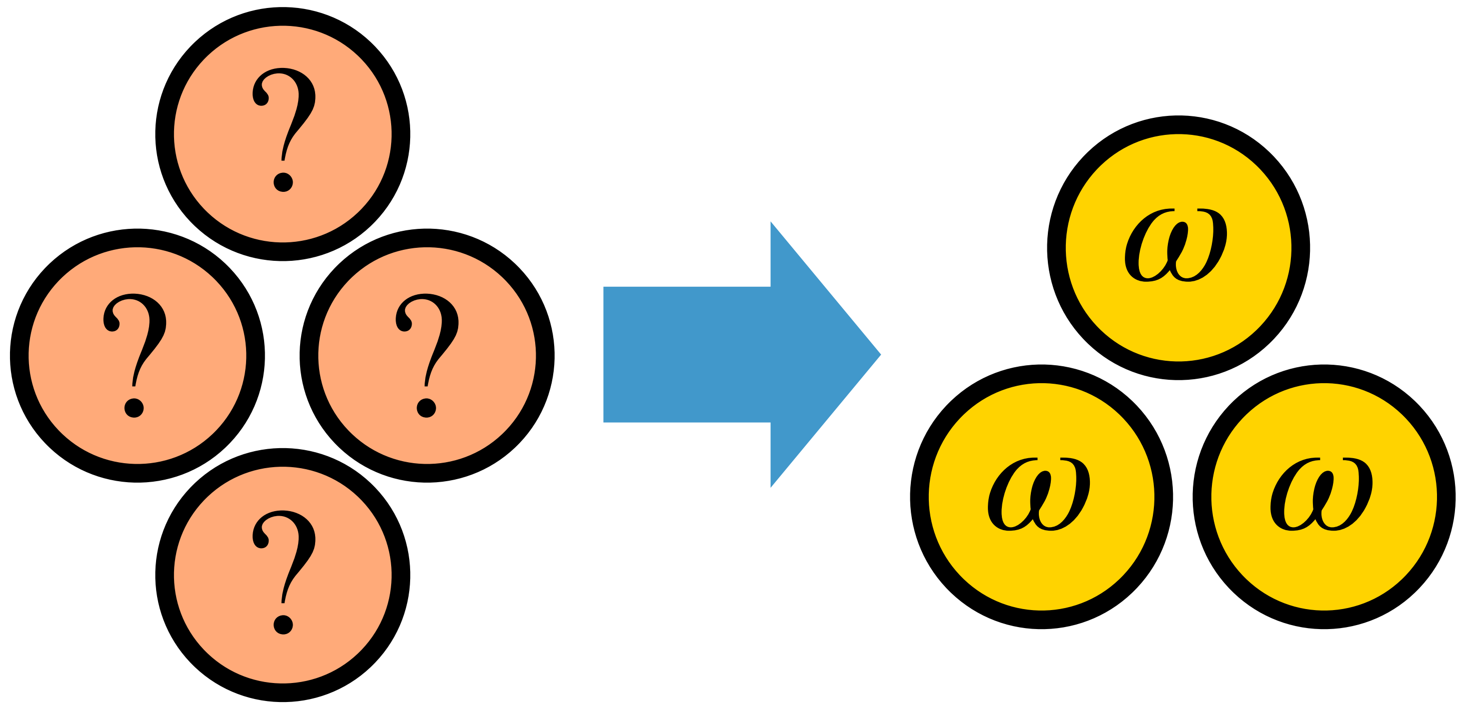

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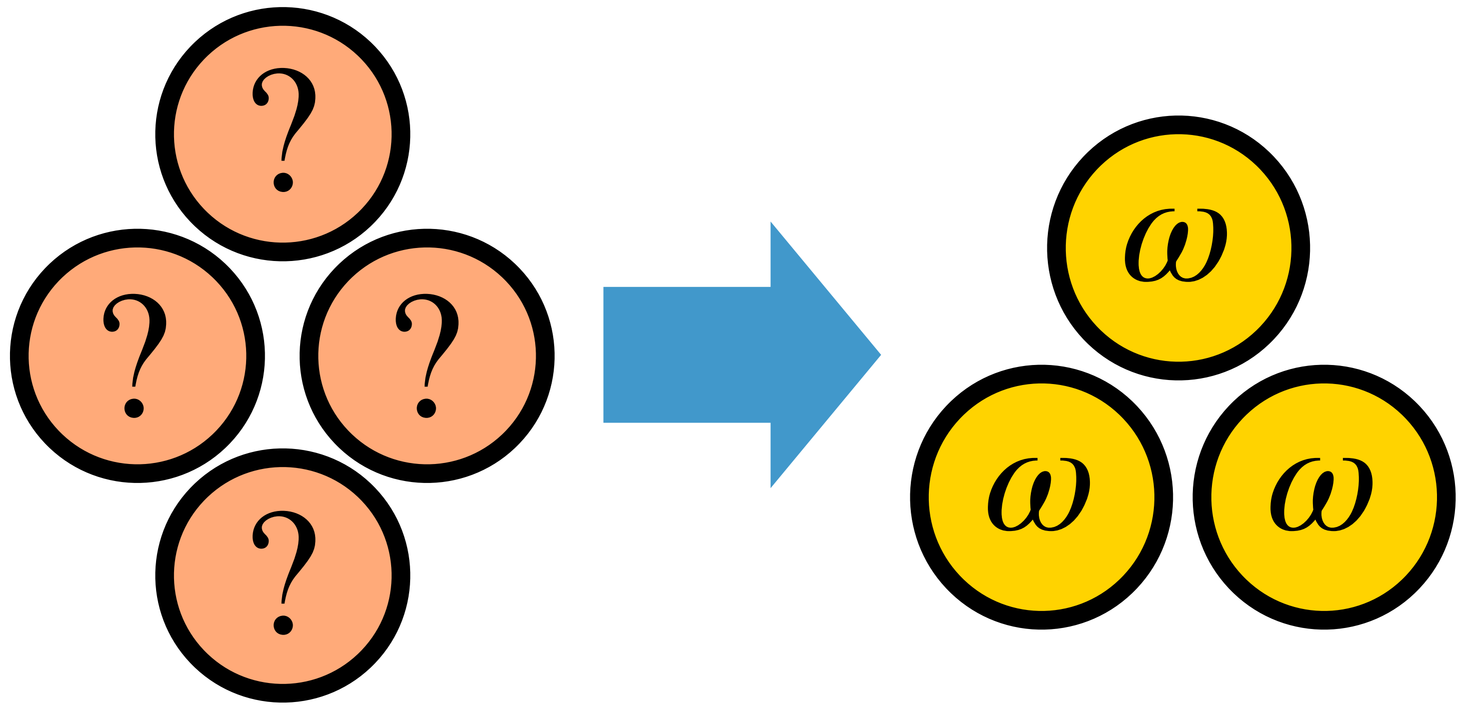

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trace norm ball of radius δ around the true state ρ

This shows that:

$$R^{\mathrm{u}}_{\mathrm{ARNG}}(\rho \to \omega) \;\geq\; \lim_{\delta \to 0^+} \frac{\mathrm{Stein}\big(B_\delta(\rho)^{\mathrm{iid}} \,\|\, \mathscr{F}\big)}{D^\infty(\omega \| \mathscr{F})}$$

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Main result and proof ideas

Theorem (Composite IID Generalised quantum Stein's lemma).

If $\mathscr{F}$ satisfies the "Brandão/Plenio axioms" and there is a topographically complete set of $\mathscr{F}$-compatible measurements (e.g. if $\mathscr{F} = \mathrm{SEP}$) then

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Strong converse holds automatically!

$\Longrightarrow$ Universal resource manipulation is possible without rate loss!

One-shot relative entropies:

- Hypothesis testing relative entropy:

$$\rho \text{ vs } \sigma : \quad -\log \min \left\{ \Pr\{\text{type 2}\} \; : \; \Pr\{\text{type 1}\} \leq \varepsilon \right\} =: D_H^{\varepsilon}(\rho \| \sigma)$$

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- Smooth max-relative entropy:

purified distance

$$D_{\max}^\varepsilon(\rho\|\sigma) := \inf \left\{ \lambda : \ \rho' \leq 2^\lambda \sigma, \ \ P(\rho, \rho') \leq \varepsilon \right\}$$

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$F(\varepsilon,\delta)$, $G(\varepsilon,\delta)$: universal functions (don't depend on the state / dimension / p)

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Not quite… The proof uses the weak/strong converse duality instead!

Proof of the composite IID GQSL. The difficult part is achievability:

$$\mathrm{Stein}\big(\mathscr{A}^{\mathrm{iid}} \,\|\, \mathscr{F}\big) = \lim_{\varepsilon\to 0^+} \liminf_{n\to\infty} \frac{1}{n} \inf_{\mu} D_H^{\varepsilon}\Big(\int_{\mathscr{A}} \mathrm{d}\mu(\rho)\ \rho^{\otimes n} \Big\| \mathscr{F}_n\Big)$$

[Fang/Fawzi/Fawzi 2024]

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Looking into the proof of GQSL in [L. 2025]

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[Fang/Fawzi/Fawzi 2024]

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Approximate quasi-concavity

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Looking into the proof of GQSL in [L. 2025]

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Can one do the same with the
Hayashi/Yamasaki proof?

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Thank you!