The Communication Power of Indefinite Causal Order

Xuanqiang Zhao

Based on joint work with Benchi Zhao, Cyril Branciard, and Giulio Chiribella

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Quantum Information and Computation Initiative

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Content

- Backgrounds on indefinite causal order
- Main results
  - Resource theory framework
  - Advantage of indefinite causal order
  - Several no-go results
- Summary and outlook

Quantum Supermap

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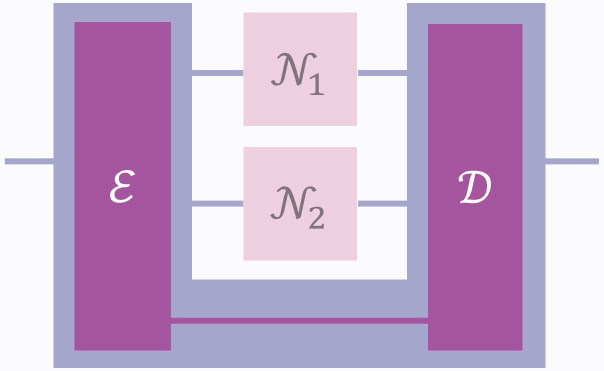
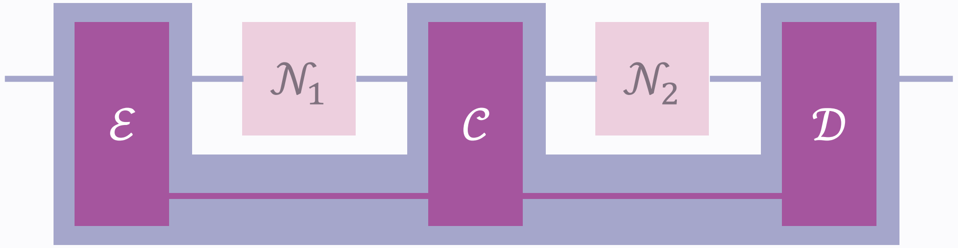
- Transformations from quantum channels to quantum channels



Parallel supermap
[Chiribella, D'Ariano, and Perinotti, EPL'08]



Sequential supermap (quantum comb)
[Chiribella, D'Ariano, and Perinotti, PRL'08]

- Both parallel and sequential supermaps are **causally definite**
- However, quantum mechanics is in principle compatible with indefinite causal order

[Hardy, J. Phys. A'07]

Indefinite Causal Order and Quantum SWITCH

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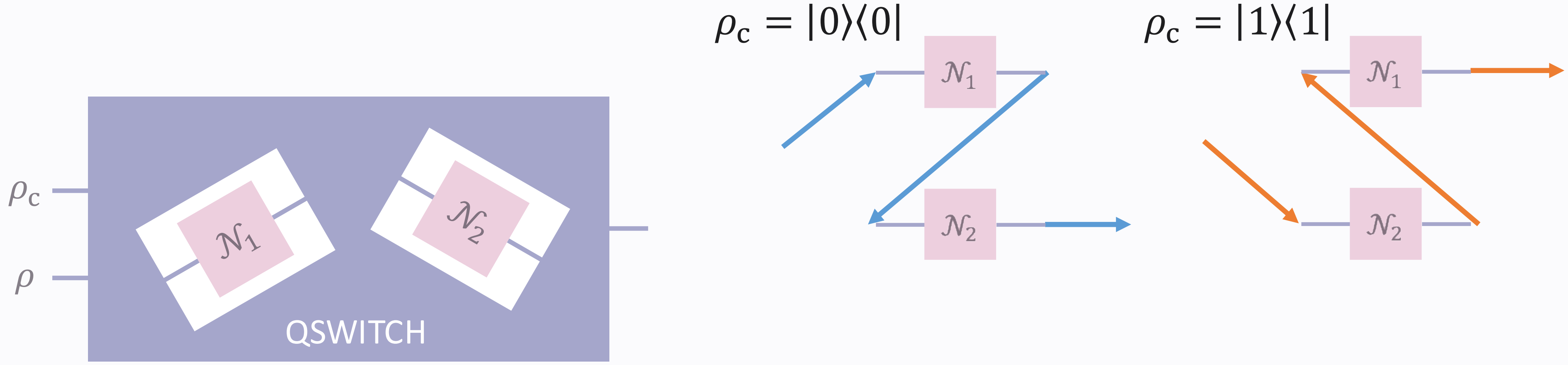
- **Quantum SWITCH** is an example of causally indefinite supermaps

  [Chiribella, D'Ariano, and Perinotti, PRA'13]



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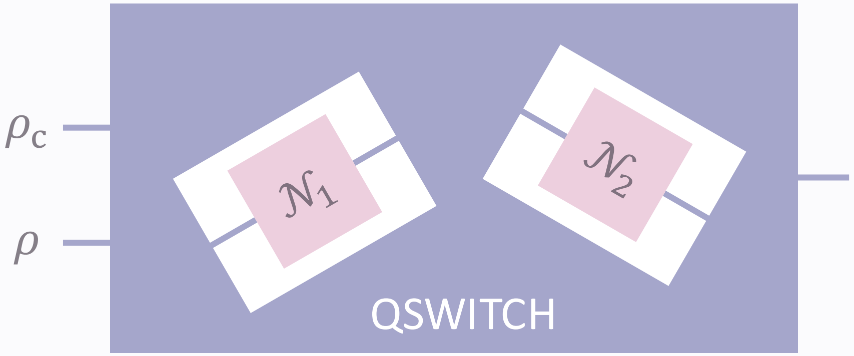
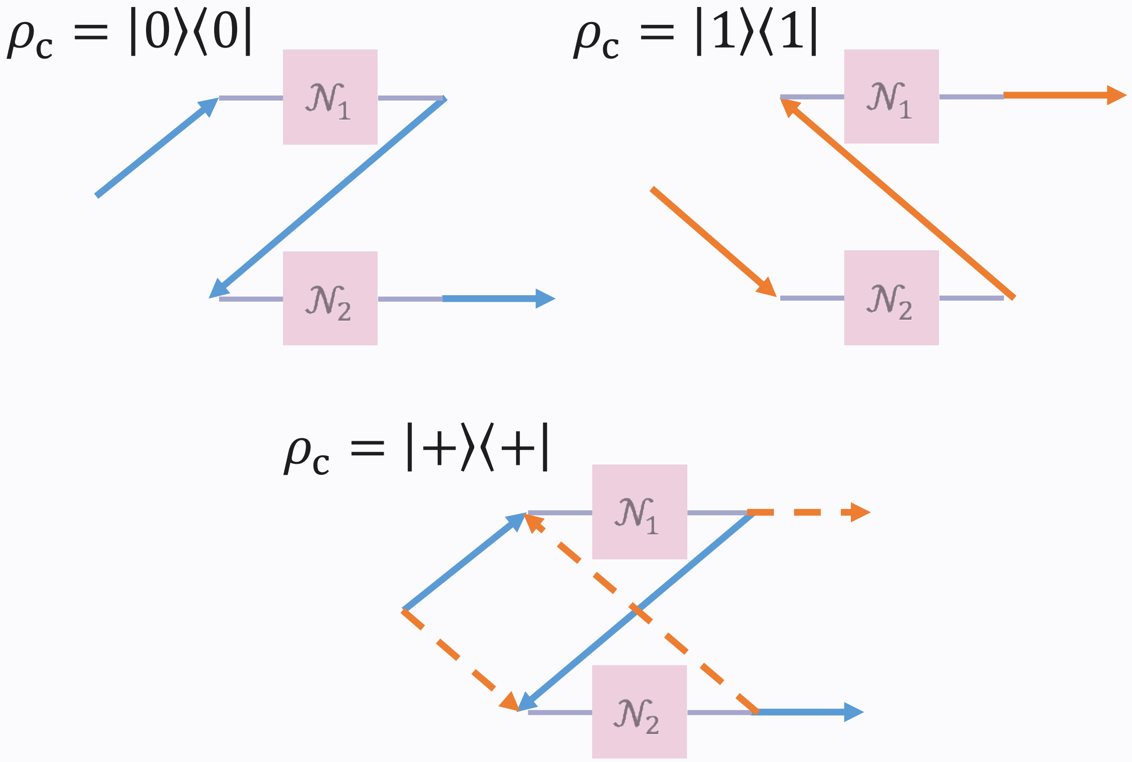




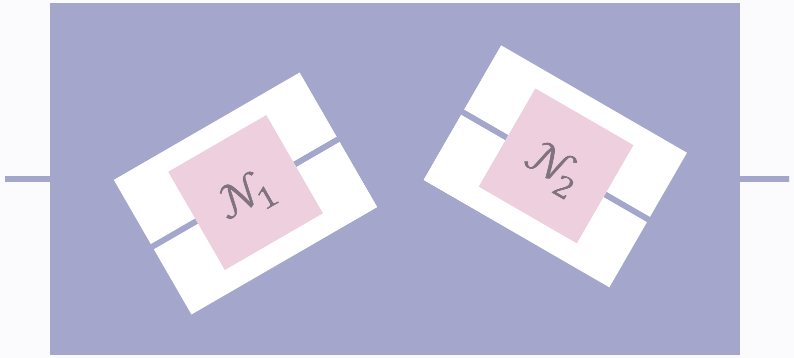
General Quantum Supermap

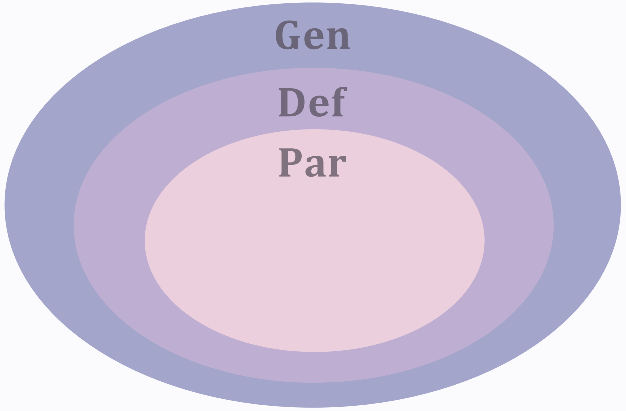
- Supermap with general, possibly indefinite, causal structure; denote the set by **Gen**

General supermap (process)

[Chiribella, D'Ariano, and Perinotti, PRA'13]

[Oreshkov, Costa, and Brukner, Nat. Comm.'12]





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Advantages of Indefinite Causal Order

- Channel discrimination

  [Chiribella, PRA'12]
  [Bavaresco, Murao, and Quintino, PRL'21]

- Quantum metrology

  [Xiaobin Zhao, Yuxiang Yang, Chiribella, PRL'20]
  [Peng Yin et al., Nat Phys'23]
  [Qiushi Liu et al., PRL'23]

- Communication complexity

  [Guérin et al., PRL'16]
  [Kejin Wei, et al., PRL'19]

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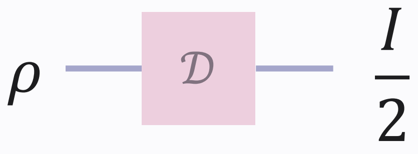
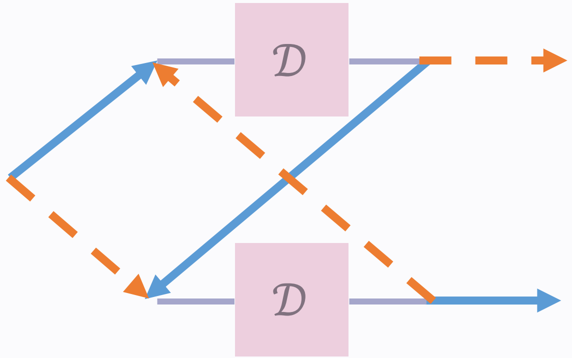
- Communication? Enhancement by QSWITCH [Ebler, Salek, and Chiribella, PRL'18]

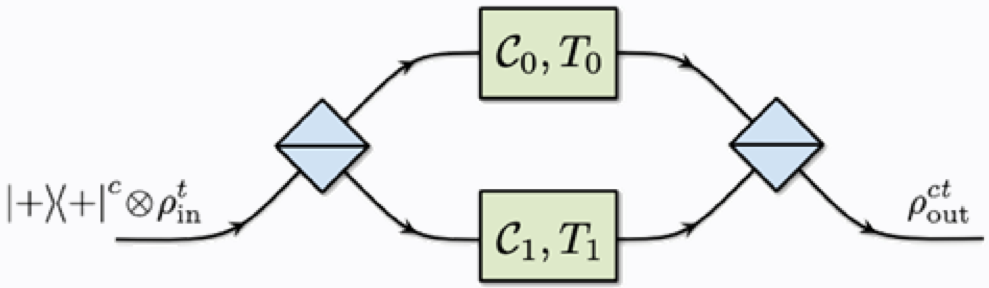
Completely depolarizing channel





Enhanced Communication by Quantum SWITCH

- Similar effect can be achieved through **coherent control of path**
  [Abbott et al., Quantum’20]



- Enhancement by QSWITCH can be matched by superposition of direct processes with the **same causal order**
  [Guérin et al., PRA’19]

- Does indefinite causal order provide any unique advantage in communication?

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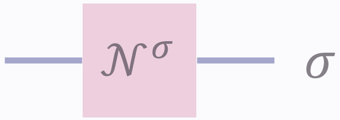
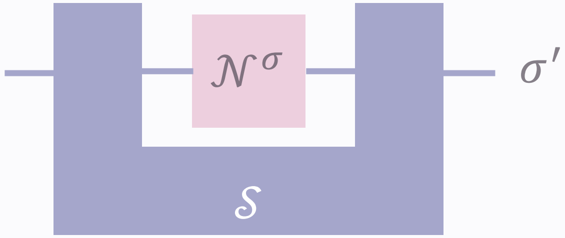
Resource Theory of Signaling

- Free channels: replacement channels



$$\mathcal{N}^{\sigma}(\cdot) \coloneqq \mathrm{Tr}[\cdot]\sigma$$

- Free supermaps: mapping free channels to free channels (resource non-generating)



$$\mathcal{S}(\mathcal{N}^{\sigma}) = \mathcal{N}^{\sigma'}, \mathcal{S} \in \mathbf{Par}$$

[Takagi, Kun Wang, and Hayashi, PRL'20]
[Kun Fang, Xin Wang, Tomamichel, and Berta, IEEE TIT'20]

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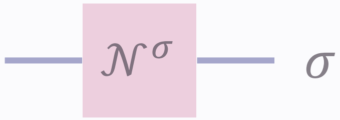

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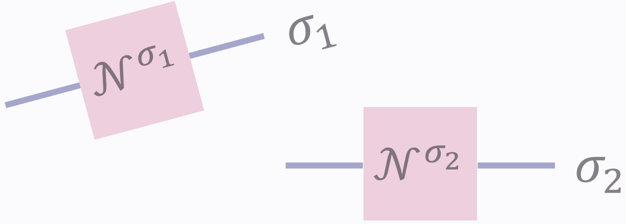

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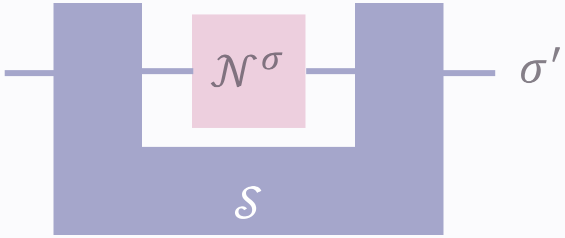

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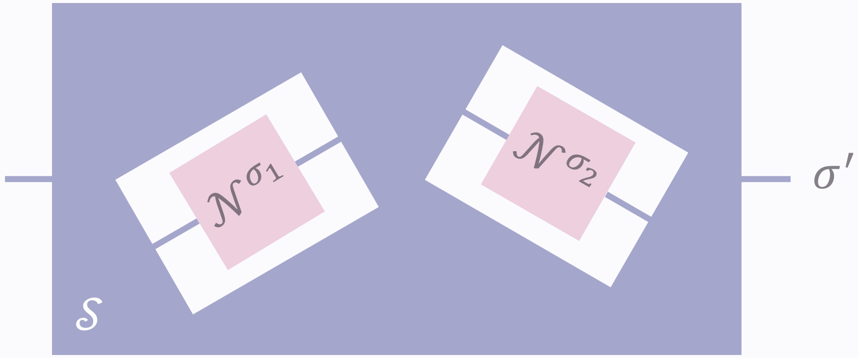
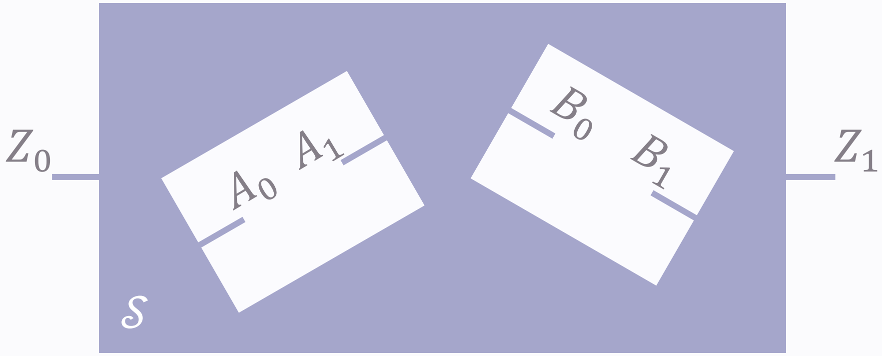


$$\mathcal{S} \in \mathbf{Gen}$$

Forward No-Signaling Supermaps

Proposition 1 *A general supermap $\mathcal{S}$ is free if and only if it is forward no-signaling.*

- **Forward no-signaling** means Z_0 cannot signal to Z_1



$$\mathrm{Tr}_{A_0B_0}\left[\mathcal{S}\left(\rho_{Z_0}\otimes\sigma_{A_1B_1}\right)\right]=\mathrm{Tr}_{A_0B_0}\left[\mathcal{S}\left(\rho'_{Z_0}\otimes\sigma_{A_1B_1}\right)\right]$$

for any states ρ_{Z_0}, ρ'_{Z_0}, and $\sigma_{A_1B_1}$

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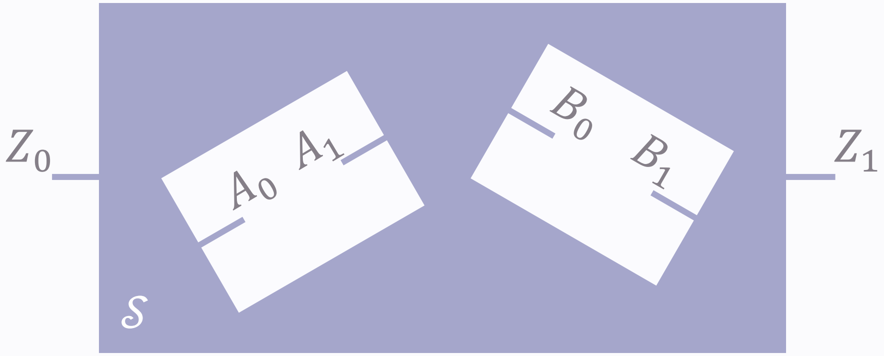

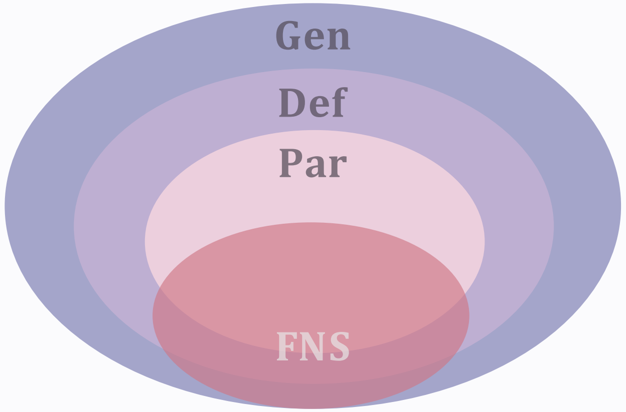

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Capacities Assisted by Different Causal Orders

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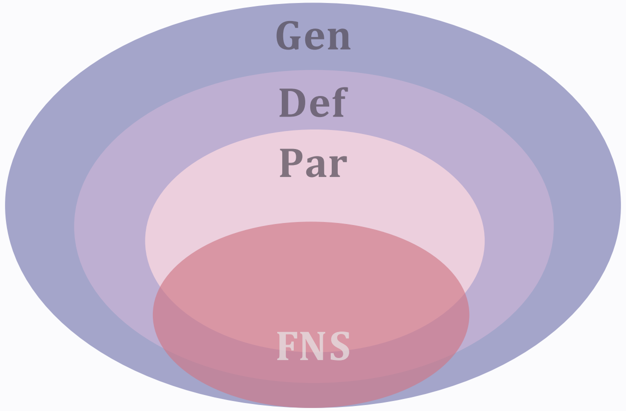

$\Omega \in \{\mathbf{Gen}, \mathbf{Def}, \dots\}$

- One-shot classical capacity assisted by free supermaps in Ω

$$C^{(1)}_{\Omega,\varepsilon}(\vec{\mathcal{N}}) := \log_2 \max\left\{m \,\middle|\, \left\|\mathcal{S}(\vec{\mathcal{N}}) - \Delta^m\right\|_\diamond \le 2\varepsilon, \mathcal{S} \in \mathbf{FNS} \cap \Omega\right\}$$

$$C^{(1)}_{\mathbf{Gen},\varepsilon}(\vec{\mathcal{N}}) \ge C^{(1)}_{\mathbf{Def},\varepsilon}(\vec{\mathcal{N}})$$

Capacities Assisted by Different Causal Orders

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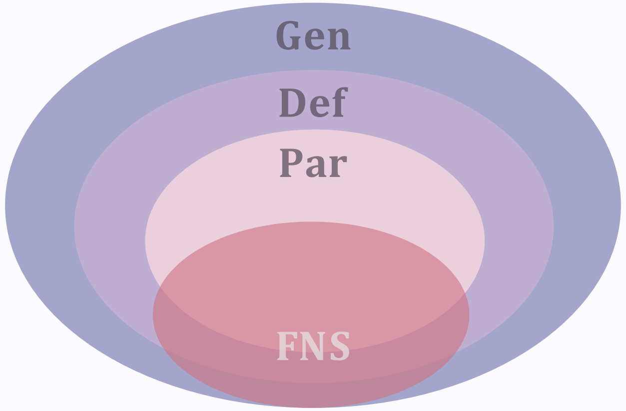

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$$C^{(1)}_{\mathbf{Gen},\varepsilon}(\vec{\mathcal{N}}) > C^{(1)}_{\mathbf{Def},\varepsilon}(\vec{\mathcal{N}}) \text{ ?}$$

Advantage in Communication

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- There exist channels $\vec{\mathcal{N}}$ such that $C^{(1)}_{\mathbf{Gen},\varepsilon}(\vec{\mathcal{N}}) > C^{(1)}_{\mathbf{Def},\varepsilon}(\vec{\mathcal{N}})$

Amplitude damping channel

$$\mathcal{A}^{\eta}(\cdot) \coloneqq K_0^{\eta}(\cdot)K_0^{\eta\,\dagger} + K_1^{\eta}(\cdot)K_1^{\eta\,\dagger}$$

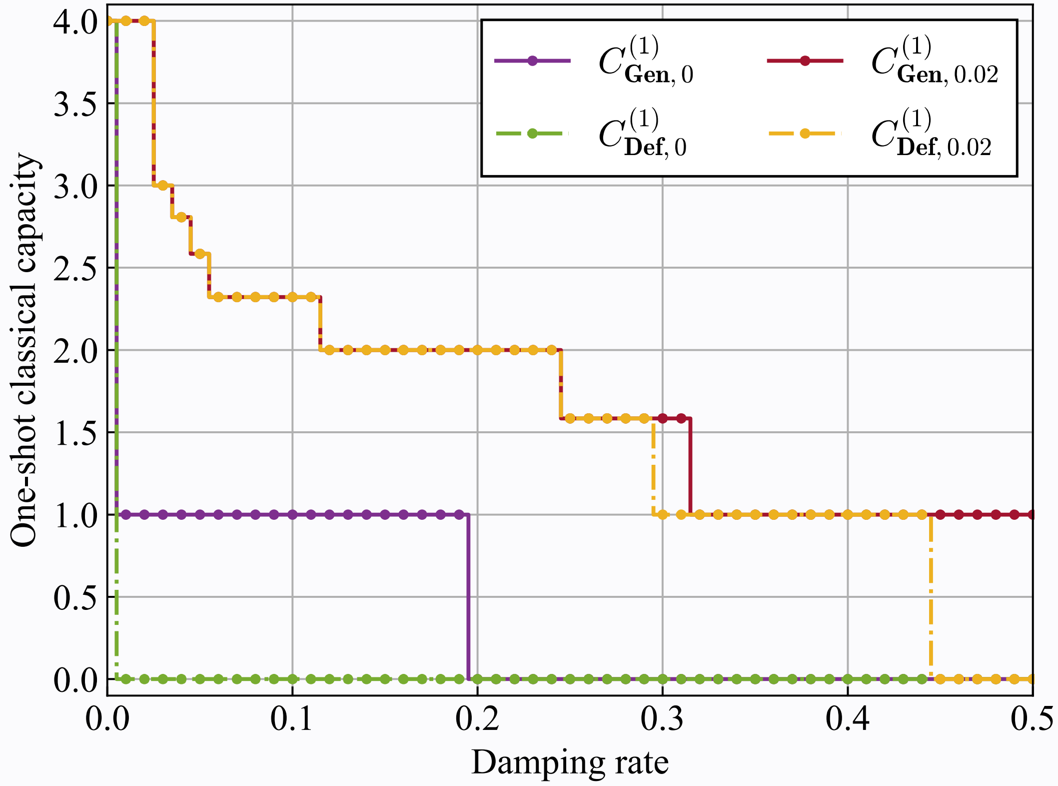
$$K_0^{\eta} \coloneqq \sqrt{\eta}|0\rangle\langle 1|,$$
$$K_1^{\eta} \coloneqq |0\rangle\langle 0| + \sqrt{1-\eta}|1\rangle\langle 1|$$

Theorem 1 $\quad C^{(1)}_{\mathbf{Gen},0}(\mathcal{A}^{0.1},\mathcal{A}^{0.1}) = 1 > C^{(1)}_{\mathbf{Def},0}(\mathcal{A}^{0.1},\mathcal{A}^{0.1}) = 0$

- Indefinite causal order enables the perfect transmission of one classical bit, while causally definite strategies fail to transmit any information

Advantage in Communication

➢ $C^{(1)}_{\mathbf{Gen},\varepsilon}(\vec{\mathcal{N}})$ and $C^{(1)}_{\mathbf{Def},\varepsilon}(\vec{\mathcal{N}})$ can be formulated as semidefinite programs



No Advantage for Multi-Qubit Pauli Channels

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- Communication over multi-qubit Pauli channels $\mathcal{P}_1, \mathcal{P}_2, \ldots$

$$\mathcal{P}(\cdot) := \sum_{\sigma \in \mathbf{P}^{\otimes q}} p_\sigma \sigma(\cdot)\sigma, \text{ probability distribution } \{p_\sigma\}, \mathbf{P} = \{I, X, Y, Z\}$$

Theorem 2 $\quad C^{(1)}_{\mathbf{Gen},\varepsilon}(\mathcal{P}_1, \mathcal{P}_2, \ldots) = C^{(1)}_{\mathbf{Par},\varepsilon}(\mathcal{P}_1, \mathcal{P}_2, \ldots)$

- Causally indefinite strategies provide no advantage over parallel strategies in the many-copy scenario

Channel Formation with Indefinite Causal Order

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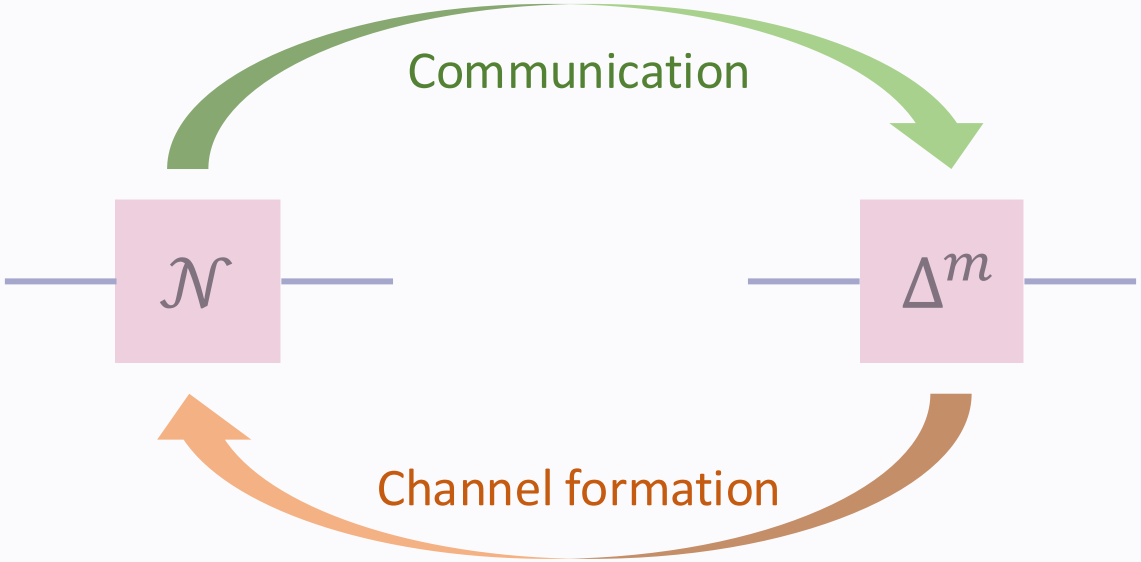
- Channel formation is the dual task of communication



- One-shot classical simulation cost assisted by free supermaps in Ω

$$S^{(1)}_{\Omega,\varepsilon}(\mathcal{N}) \coloneqq \log_2 \min\{m \mid \|\mathcal{S}(\Delta^m) - \mathcal{N}\|_\diamond \le 2\varepsilon, \mathcal{S} \in \mathbf{FNS} \cap \Omega\}$$

Channel Formation with Indefinite Causal Order

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- It is natural to expect that indefinite causal order also provide advantage in this dual task

$$S^{(1)}_{\mathbf{Gen},\varepsilon}(\mathcal{N}) < S^{(1)}_{\mathbf{Def},\varepsilon}(\mathcal{N}) \text{ ?}$$

- Surprisingly, indefinite causal order performs no better than parallel strategies

Theorem 3 $S^{(1)}_{\mathbf{Gen},\varepsilon}(\mathcal{N}) = S^{(1)}_{\mathbf{Par},\varepsilon}(\mathcal{N})$

Asymptotic Limit

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$$C_\Omega(\mathcal{N}) \coloneqq \lim_{\varepsilon\to 0}\lim_{N\to\infty}\frac{1}{N}C^{(1)}_{\Omega,\varepsilon}\left(\mathcal{N}^{\otimes N}\right) \qquad S_\Omega(\mathcal{N}) \coloneqq \lim_{\varepsilon\to 0}\lim_{N\to\infty}\frac{1}{N}S^{(1)}_{\Omega,\varepsilon}\left(\mathcal{N}^{\otimes N}\right)$$

- Compare with entanglement-assisted capacity/simulation cost

$$C_{\mathrm{E}}(\mathcal{N}) \le C_{\mathbf{Gen}}(\mathcal{N}) \qquad S_{\mathbf{Gen}}(\mathcal{N}) \le S_{\mathrm{E}}(\mathcal{N})$$

Entanglement-assisted classical capacity

Entanglement-assisted classical simulation cost

Asymptotic Limit

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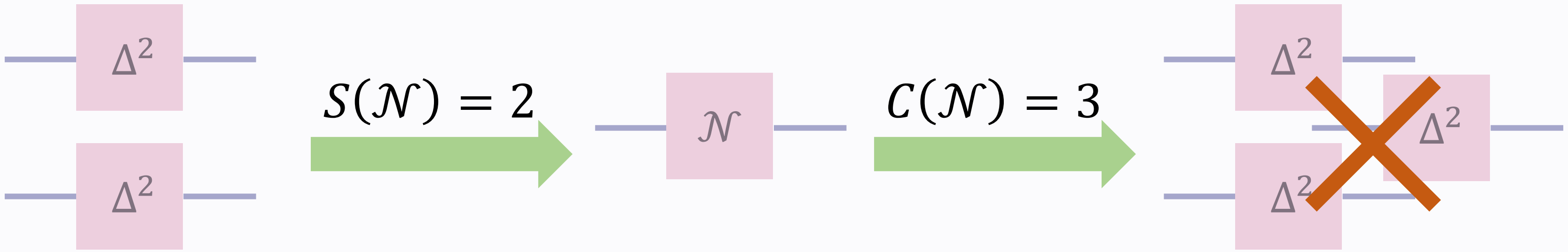

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$$C_{\mathrm{E}}(\mathcal{N}) \le C_{\mathbf{Gen}}(\mathcal{N}) \le S_{\mathbf{Gen}}(\mathcal{N}) \le S_{\mathrm{E}}(\mathcal{N})$$

- Quantum reverse Shannon theorem: $C_{\mathrm{E}}(\mathcal{N}) = S_{\mathrm{E}}(\mathcal{N})$

[Bennett et al., IEEE TIT'14]

[Berta, Christandl, and Renner, CMP'11]

$$C_{\mathrm{E}}(\mathcal{N}) = C_{\mathbf{Gen}}(\mathcal{N}) = S_{\mathbf{Gen}}(\mathcal{N}) = S_{\mathrm{E}}(\mathcal{N})$$

Summary and Outlook

- We establish **a resource theory of communication** allowing indefinite causal order as a framework for investigating the potential advantage of indefinite causal order
- We demonstrate the **advantage of indefinite causal order in communication** with amplitude damping channels; for multi-qubit Pauli channels, parallel strategies achieve optimal performance
- Unlike in communication, indefinite causal order offers **no advantage in channel formation**
- In the **asymptotic limit, no advantage** due to the quantum reverse Shannon theorem

❖ Take a similar approach in other quantum resource theories, e.g., quantum thermodynamics and nonstabilizerness [Yin Mo et al., PRA'24] [Felce and Vedral, PRL'20]

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Thank you for your attention!