Reliability of asymptotic work extraction

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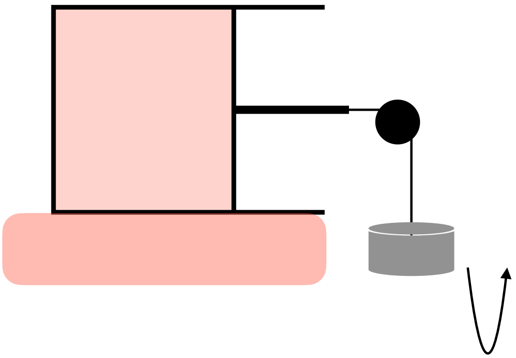
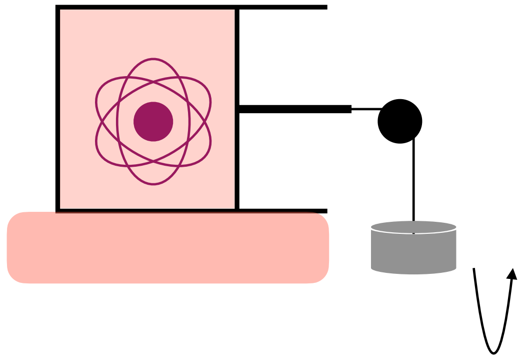
Quantum thermodynamics

Macroscopic thermodynamics



Second law of thermodynamics

quantum thermodynamics



Second law of **quantum** thermodynamics

- We mainly consider the **isothermal process**

  (Finite-dimensional system, β : inverse temperature of the heat bath)

  H: Hamiltonian of the system, $\tau = e^{-\beta H}/\mathrm{Tr}\ e^{-\beta H}$: Gibbs thermal state

- **Isothermal operations**?

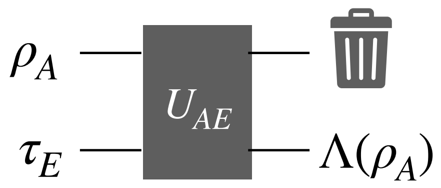
Isothermal operations

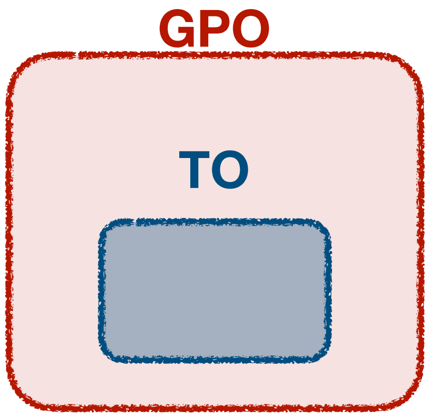
[Horodecki, Oppenheim, Nat. Commun.(2019)],[Faist, Renner, PRX (2018)]

- **Thermal operations** (operationally motivated class) [Horodecki, Oppenheim, Nat. Commun.(2019)],

$$\Lambda \in \mathrm{TO} \Leftrightarrow \Lambda(\rho_A) = \mathrm{Tr}_{AB\setminus E}[U_{AE}(\rho_A \otimes \tau_E)U_{AE}^\dagger]$$

($[U_{AE}, H_A \otimes I_E + I_A \otimes H_E] = 0$: energy-conservation)





Properties:
- $\Lambda(\tau_A) = \tau_B$ (Gibbs-preserving condition)
- $\Lambda(e^{-iH_At}\rho e^{iH_At}) = e^{-iH_Bt}\Lambda(\rho)e^{iH_Bt}$ (time-translation covariance)

Time-translation covariance (=energy conservation) prohibits one from creating/detecting coherence

- **Gibbs-preserving operations** (mathematically simple class)

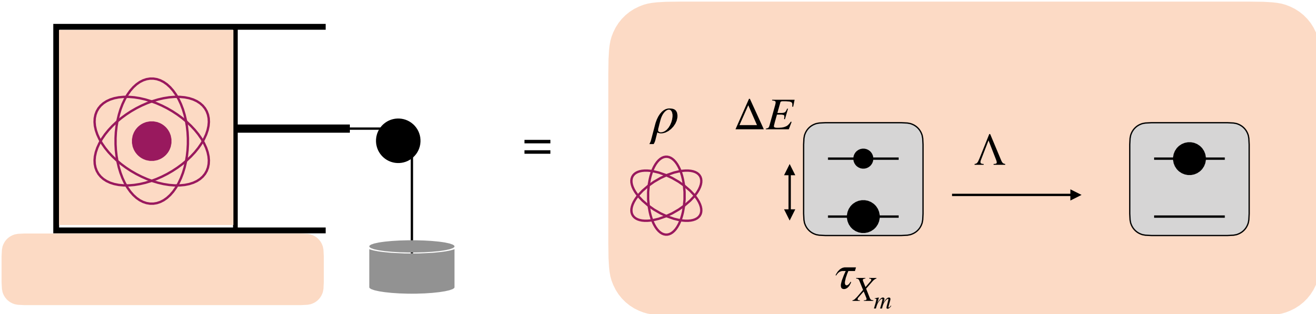
  Includes all channels satisfying Gibbs-preserving condition
- **Thermal operations** $\subsetneq$ **Gibbs-preserving operations**
- The framework can be seen as a **resource theory of thermodynamics**

  Free operations: either TO or GPO, Free state: τ

Work extraction

[Horodecki, Oppenheim, Nat. Commun. (2013)]

[Brandão et al., PRL (2013)]



Work battery (labeled by $m > 1$)

$\mathscr{H}_{X_m} = \mathrm{Span}\{|0\rangle, |1\rangle\}$

$H_{X_m} = 0|0\rangle\langle 0| + \beta^{-1}\log(m-1)|1\rangle\langle 1|$

$\tau_{X_m} = \frac{1}{m}|1\rangle\langle 1| + \frac{m-1}{m}|0\rangle\langle 0|$

- Work $\beta W = \log m$ is extracted $\Leftrightarrow$ $\exists \Lambda \in$ (GPO or TO) such that $\rho \otimes \tau_{X_m} \xrightarrow{\Lambda} |1\rangle\langle 1|_{X_m}$

- **One-shot extractable work from ρ under a class of free operations $\mathbb{O}$ with target error ε**

$$\beta W_{\mathbb{O}}^{\varepsilon}(\rho) := \log \max\{m \mid \max_{\Lambda \in \mathbb{O}} \langle 1 | \Lambda(\rho \otimes \tau_{X_m}) | 1 \rangle \geq 1 - \varepsilon\}$$

- **Asymptotic extractable work rate from $(\rho^{\otimes n})_{n \in \mathbb{N}}$ under a class of free operations $\mathbb{O}$**

$$\beta W_{\mathbb{O}}^{\mathrm{asymp}}(\rho) := \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \beta W_{\mathbb{O}}^{\varepsilon}(\rho^{\otimes n})$$

Work extraction in asymptotic limit

Previous results

For both the class of **thermal operations** and **Gibbs-preserving operations**

$$\beta W_{\mathrm{TO}}^{\mathrm{asymp}}(\rho) = \beta W_{\mathrm{GPO}}^{\mathrm{asymp}}(\rho) = D(\rho\|\tau) = F(\rho) - F_{\mathrm{eq}}$$

$$D(\rho\|\tau) := \mathrm{Tr}[\rho \log \rho - \rho \log \tau]$$

Umegaki relative entropy

- Optimal work rate can be characterized as the difference in Helmholtz free energy (**Quantum version of Kelvin's principle**)

[Brandão et al., PRL (2013)] [Gour, PRX Quantum (2022)]

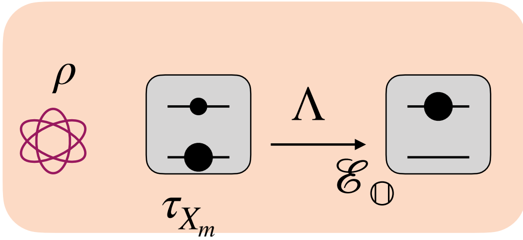
This would suggest: The choice of operation (especially, **time-translational covariance**) is not important in the asymptotic work extraction

Our Question: Isn't time-translation covariance important?

Analysis from <u>a refined perspective</u>: **Reliability (= Optimal error decay) of work extraction**

Optimal error/ reliability of work extraction

We will focus on **the smallest error** in the final state of the work battery when **we fix the target work**



Def: One-shot optimal error of work extraction from ρ with target amount of work W

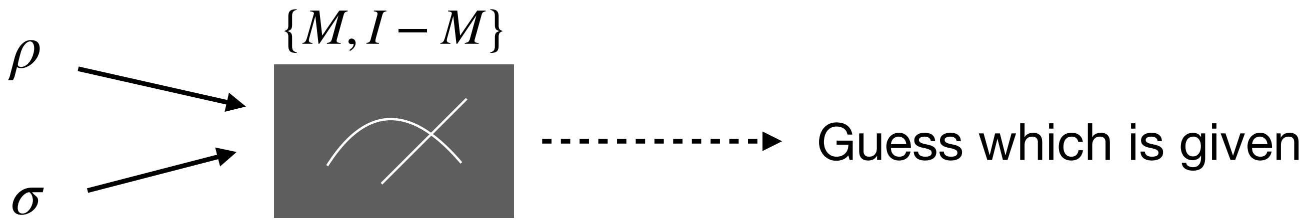
$$\mathscr{E}_{\mathbb{O}}(\rho;\beta W) := 1 - \max_{\Lambda\in\mathbb{O}} F(\Lambda(\rho\otimes\tau_{X_{e^{\beta W}}}), |1\rangle\langle 1|_{X_{e^{\beta W}}})$$

- In the asymptotic scenario, when the rate r is suboptimal, $\beta W_n \sim rn,\ \varepsilon_n \sim e^{-cn}$

Def: Reliability(=error exponent) of work extraction with a target work extraction rate r

$$\mathrm{Rel}_{\mathbb{O}}(\rho; r) := \sup_{\{W_n\}_n} \left\{ \liminf_{n\to\infty} -\frac{1}{n}\log \mathscr{E}_{\mathbb{O}}(\rho^{\otimes n}, \beta W_n) \;\middle|\; \liminf_{n\to\infty} \frac{1}{n}\beta W_n \geq r \right\}$$

Quantum hypothesis testing



Type I error: ρ is given but guess σ is given with prob. $\alpha(\rho, M) := \mathrm{Tr}[\rho(I - M)]$

Type II error: σ is given but guess ρ is given with prob. $\beta(\sigma, M) := \mathrm{Tr}[M\sigma]$

- **Hypothesis testing divergence**

$$D_H^\varepsilon(\rho\|\sigma) := -\log \inf\{\mathrm{Tr}[M\sigma] \mid \mathrm{Tr}[(I - M)\rho] \le \varepsilon,\ 0 \le M \le I\}$$

- Asymptotic limit in the multicopy setting ($\rho^{\otimes n}$ vs $\sigma^{\otimes n}$, $n \to \infty$)

Quantum Stein's lemma (minimize type II error while type I error is kept constant)

$$\lim_{n\to\infty} \frac{1}{n} D_H^\varepsilon(\rho^{\otimes n}\|\sigma^{\otimes n}) = D(\rho\|\sigma), \quad \forall \varepsilon \in (0,1)$$

[Hiai, Petz, Comm. Mat. Phys. (1991)]
[Ogawa, Nagaoka, IEEE. Trans. Info. Theory., (2000)]

Optimal error of work extarction

Proposition (Folklore): One-shot optimal error of work extraction

The one-shot optimal error of work extraction from ρ with target work W under **Gibbs-preserving operations** and **thermal operations** are given by

$$\mathscr{E}_{\mathrm{GPO}}(\rho;\beta W) = \min\{\varepsilon \mid \beta W \leq D_H^{\varepsilon}(\rho\|\tau)\}$$

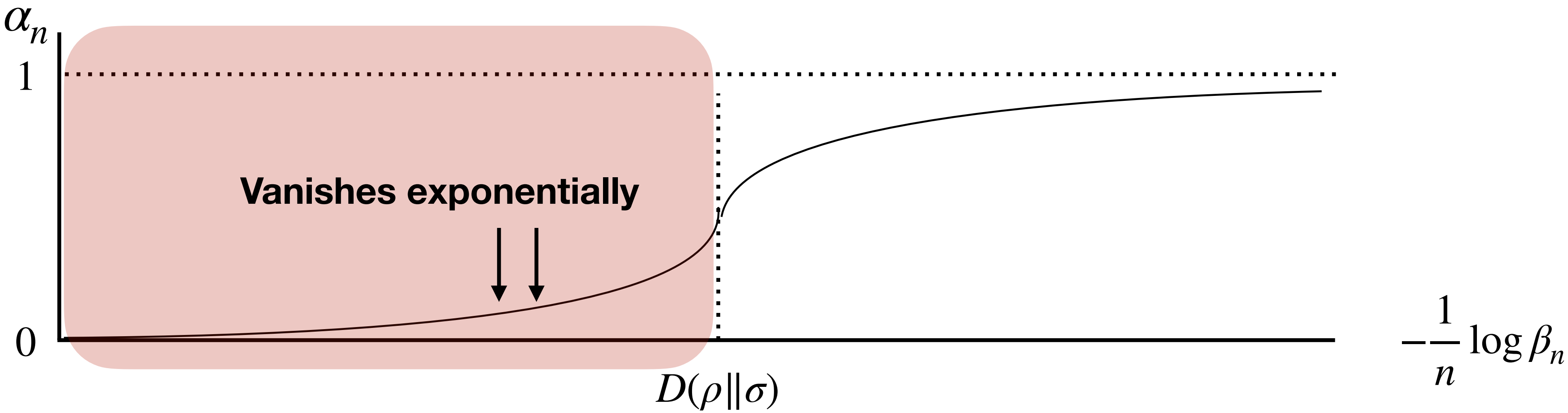
$$\mathscr{E}_{\mathrm{TO}}(\rho;\beta W) = \min\{\varepsilon \mid \beta W \leq D_H^{\varepsilon}(\mathscr{P}_\tau(\rho)\|\tau)\}$$

$\mathscr{P}_\tau(\,\cdot\,) := \sum_i \Pi_i(\,\cdot\,)\Pi_i$:spectral pinching **(Time-average)** $\quad$ ($\tau = \sum_i \lambda_i \Pi_i$ spectral decomposition)

- Pinching $\mathscr{P}_\tau(\,\cdot\,)$ in $\mathscr{E}_{\mathrm{TO}}(\rho;\beta W)$ represents the restriction due to the **time-translation covariance**
- $\mathscr{E}_{\mathrm{GPO}}(\rho;\beta W) \leq \mathscr{E}_{\mathrm{TO}}(\rho;\beta W)$, due to DPI of hypothesis testing divergence

The one-shot expression enables us to connect the reliability with the exponents of hypothesis testing

Error exponent of hypothesis testing



Error exponent

$$B_H((\rho^{\otimes n})_n\|(\sigma^{\otimes n})_n; r) := \sup\left\{\liminf_{n\to\infty} -\frac{1}{n}\log\alpha(\rho^{\otimes n}, M_n) \;\middle|\; \liminf_{n\to\infty} -\frac{1}{n}\log\beta(\sigma^{\otimes n}, M_n) \geq r\right\}$$

Fact: [Nagaoka (2006), Hayashi, PRA(2007)]

$$B_H((\rho^{\otimes n})_n\|(\sigma^{\otimes n})_n; r) = \sup_{0<\alpha<1}\frac{\alpha-1}{\alpha}(r - \overline{D}_\alpha(\rho\|\sigma))$$

$$\overline{D}_\alpha(\rho\|\sigma) := \frac{1}{\alpha-1}\log\mathrm{Tr}[\rho^\alpha\sigma^{1-\alpha}]$$

Petz Rényi relative entropy

Reliability of work extraction

$$\widetilde{D}_\alpha(\rho\|\sigma) := \frac{1}{\alpha-1}\log\mathrm{Tr}\left[\left(\sigma^{\frac{1-\alpha}{2\alpha}}\rho\sigma^{\frac{1-\alpha}{2\alpha}}\right)^\alpha\right]$$

Sandwiched Rényi relative entropy

Theorem : Reliability of work extraction

- Reliability of asymptotic work extraction with a target work rate r is characterized as

$$\mathrm{Rel}_{\mathrm{GPO}}(\rho;r) = B_H((\rho^{\otimes n})_n\|(\tau^{\otimes n})_n, r) = \sup_{0<\alpha<1}\frac{\alpha-1}{\alpha}\left(r-\overline{D}_\alpha(\rho\|\tau)\right)$$

$$\mathrm{Rel}_{\mathrm{TO}}(\rho;r) = B_H((\mathscr{P}_{\tau^{\otimes n}}(\rho^{\otimes n}))_n\|(\tau^{\otimes n})_n, r)$$

$\mathrm{Rel}_{\mathrm{GPO}}(\rho;r) > \mathrm{Rel}_{\mathrm{TO}}(\rho;r)$ holds for any $\forall r\in(0,\,D(\rho\|\tau))$ whenever $[\rho,H]\neq 0$.

- The separation in reliability is sharply represented as the difference of divergences ($\overline{D}_\alpha(\rho\|\tau)\geq\widetilde{D}_\alpha(\rho\|\tau)$) (strict separation comes from the equality condition of ALT inequality) [Friedland, So (1994)]
- Operational interpretation of $\widetilde{D}_\alpha$ of order $\alpha\in(0,1)$, which arises from **time-translation covariance**

Zero-rate error exponent

Q. How fast does the error decay if you are satisfied with an $O(1)$ amount of work?

Def: Zero-rate error exponent of work extraction from ρ

$$\mathrm{Rel}^{\mathrm{Z.R.}}_{\mathbb{O}}(\rho) := \lim_{W\to\infty} \liminf_{n\to\infty} -\frac{1}{n}\log \mathscr{E}_{\mathbb{O}}(\rho^{\otimes n};\beta W)$$

Theorem

$$\mathrm{Rel}^{\mathrm{Z.R.}}_{\mathrm{GPO}}(\rho) = D(\tau\|\rho) \qquad \mathrm{Rel}^{\mathrm{Z.R.}}_{\mathrm{TO}}(\rho) = D^{\star}(\tau\|\rho)$$

$$D^{\star}(\sigma\|\rho) := \lim_{\alpha\to 1}\lim_{n\to\infty}\frac{1}{n}D_\alpha(\sigma^{\otimes n}\|\mathscr{P}_{\sigma^{\otimes n}}(\rho^{\otimes n})) = \lim_{\alpha\to 0}\frac{1-\alpha}{\alpha}\widetilde{D}_\alpha(\rho\|\sigma)$$: **Star divergence**

(Equivalently, $\alpha\to 1$ limit of the reverse sandwiched Rényi divergence $\widehat{D}_\alpha(\rho\|\sigma) = \frac{\alpha}{1-\alpha}\widetilde{D}_\alpha(\sigma\|\rho)$)

From DPI of Petz/sandwiched Rényi divergences, we have $D(\tau\|\rho) \geq D^{\star}(\tau\|\rho)$

Data-processing inequality under covariance

$$\mathrm{Rel}_{\mathrm{TO}}(\rho; r) = \sup_{0<\alpha<1} \frac{\alpha-1}{\alpha}\left(r - \widetilde{D}_\alpha(\rho\|\tau)\right) \qquad \mathrm{Rel}_{\mathrm{TO}}^{\mathrm{Z.R.}}(\rho) = D^\star(\tau\|\rho) = \lim_{\alpha\to 0}\frac{1-\alpha}{\alpha}\widetilde{D}_\alpha(\rho\|\tau)$$

- Sandwiched Rényi divergence satisfies DPI for $\alpha \geq \frac{1}{2}$: $\widetilde{D}_\alpha(\rho\|\sigma) \geq \widetilde{D}_\alpha\left(\Lambda(\rho)\|\Lambda(\sigma)\right)$ ($\forall\Lambda$:CPTP)
- However, the sandwiched Rényi divergence does not satisfy DPI for $\alpha \in \left(0, \frac{1}{2}\right)$.
- The star divergence $D^\star(\tau\|\rho)$ does not satidfy DPI as well.

[Berta, Fawzi, Tomamichel, LMP (2017)]

Proposition

Let σ be a quantum state. For a channel Λ satisfying $\mathscr{P}_{\Lambda(\sigma)^{\otimes n}} \circ \Lambda^{\otimes n} \circ \mathscr{P}_{\sigma^{\otimes n}} = \mathscr{P}_{\Lambda(\sigma)^{\otimes n}} \circ \Lambda^{\otimes n}$, we have

$$\widetilde{D}_\alpha(\rho\|\sigma) \geq \widetilde{D}_\alpha\left(\Lambda(\rho)\|\Lambda(\sigma)\right), \quad \forall\alpha \geq 0 \qquad D^\star(\sigma\|\rho) \geq D^\star\left(\Lambda(\sigma)\|\Lambda(\rho)\right)$$

- When we take $\sigma = \tau$ (thermal state), thermal operations satisfy the covariance conditions

Case studies

Reliability under GPO and TO in several situations

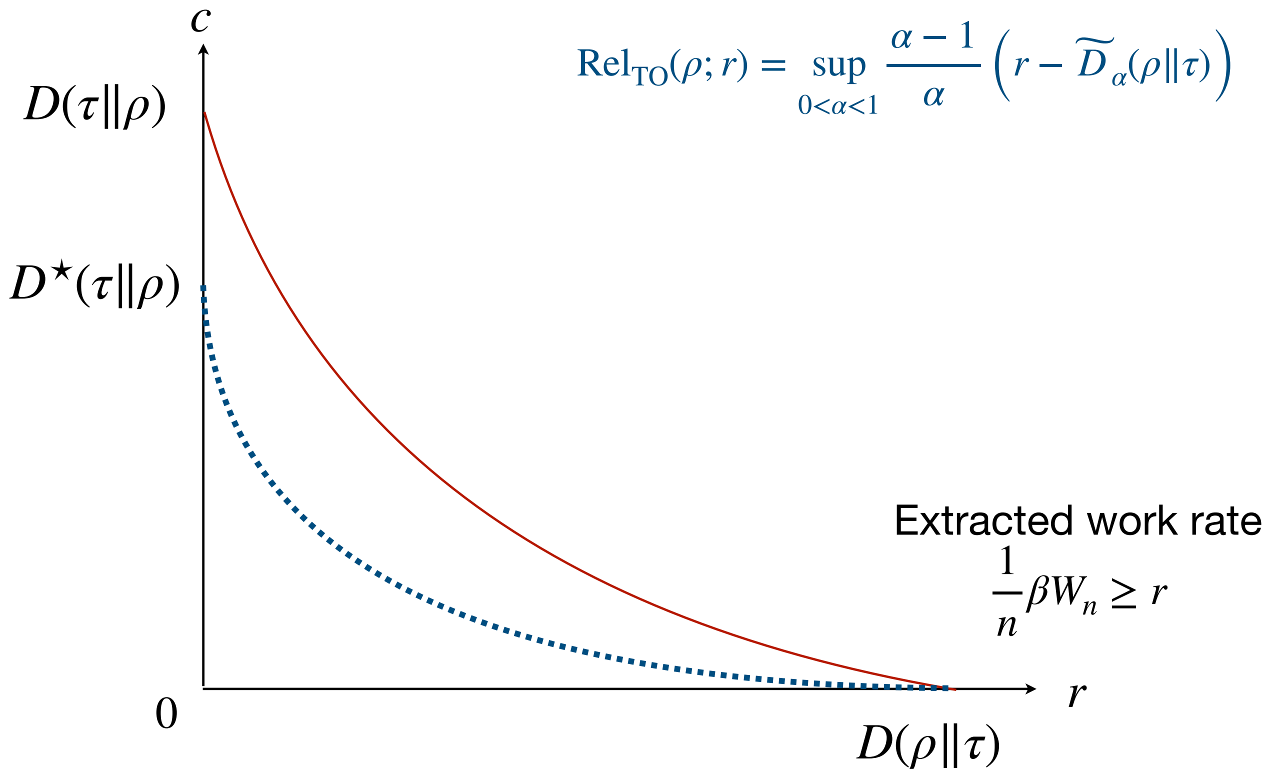
Reliability: $[\rho, H] \neq 0$ and full-rank ρ

Error decay (Reliability)

$$\varepsilon_n \sim 2^{-cn}$$

$$\mathrm{Rel}_{\mathrm{GPO}}(\rho; r) = \sup_{0<\alpha<1} \frac{\alpha - 1}{\alpha}\left(r - \overline{D}_\alpha(\rho\|\tau)\right)$$

$$\mathrm{Rel}_{\mathrm{TO}}(\rho; r) = \sup_{0<\alpha<1} \frac{\alpha - 1}{\alpha}\left(r - \widetilde{D}_\alpha(\rho\|\tau)\right)$$



$[\rho, \tau] \neq 0 \quad \Leftrightarrow \overline{D}_\alpha(\rho\|\tau) > \widetilde{D}_\alpha(\rho\|\tau)$

$\Rightarrow \mathrm{Rel}_{\mathrm{GPO}}(\rho; r) > \mathrm{Rel}_{\mathrm{TO}}(\rho; r)$

$\forall r \in (0,\ D(\rho\|\tau))$

Strict separation!

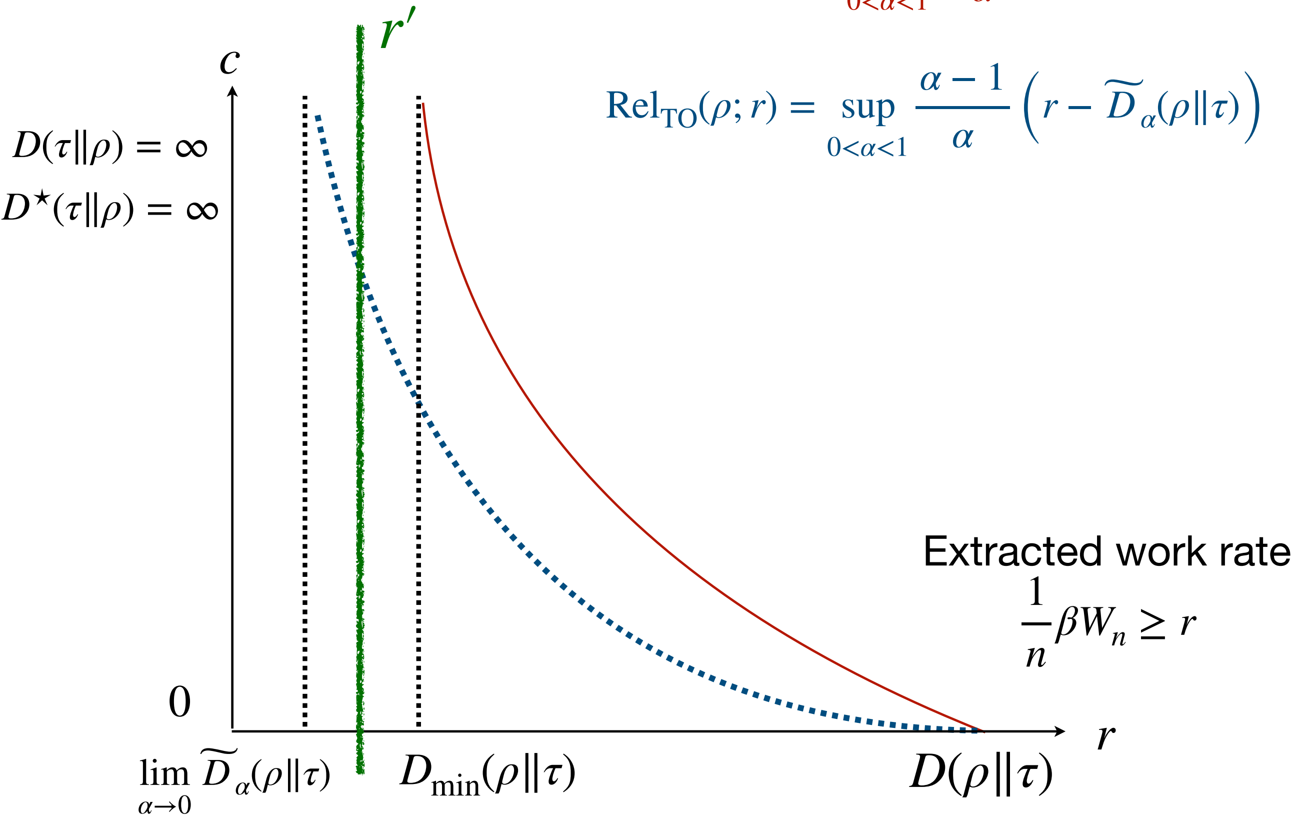
Reliability: $[\rho, H] \neq 0$ and rank-defficient ρ

Error decay (Reliability)

$\varepsilon_n \sim 2^{-cn}$

$$\mathrm{Rel}_{\mathrm{GPO}}(\rho; r) = \sup_{0<\alpha<1} \frac{\alpha - 1}{\alpha}\left(r - \overline{D}_\alpha(\rho\|\tau)\right)$$

$$\mathrm{Rel}_{\mathrm{TO}}(\rho; r) = \sup_{0<\alpha<1} \frac{\alpha - 1}{\alpha}\left(r - \widetilde{D}_\alpha(\rho\|\tau)\right)$$



$\mathrm{supp}(\tau) \not\subset \mathrm{supp}(\rho) \Rightarrow D(\tau\|\rho) = +\infty$

$\mathrm{Rel}_{\mathrm{GPO}}(\rho; r)$ diverges for $0 < r < D_{\min}(\rho\|\tau)$

$\mathrm{supp}(\tau) \not\subset \mathrm{supp}(\rho) \Rightarrow D^\star(\tau\|\rho) = +\infty$

(From the closed form in [Audenaert, Datta, JMP(2015)])

$\mathrm{Rel}_{\mathrm{TO}}(\rho; r)$ diverges for $0 < r < \lim_{\alpha\to 0} \widetilde{D}_\alpha(\rho\|\tau)$

$\lim_{\alpha\to 0} \widetilde{D}_\alpha(\rho\|\tau) < r' < D_{\min}(\rho\|\tau)$

$\Rightarrow \mathrm{Rel}_{\mathrm{GPO}}(\rho; r) = +\infty,\ \mathrm{Rel}_{\mathrm{TO}}(\rho; r) < \infty$

When does it happen?

Equality condition of $D_{\min}(\rho\|\tau) \geq \lim_{\alpha\to 0} \widetilde{D}_\alpha(\rho\|\tau)$

- An example was discussed in [Datta, Leditzky JPA(2014)]

Proposition

Let ρ, σ be two quantum states with $\mathrm{supp}(\rho) \subset \mathrm{supp}(\sigma)$.

Then, $D_{\min}(\rho\|\sigma) = \lim_{\alpha\to 0} \widetilde{D}_\alpha(\rho\|\sigma)$ holds if and only if $[\Pi_{\mathrm{supp}(\rho)}, \sigma] = 0$

This separation happens whenever $[\Pi_{\mathrm{supp}(\rho)}, \tau] \neq 0$

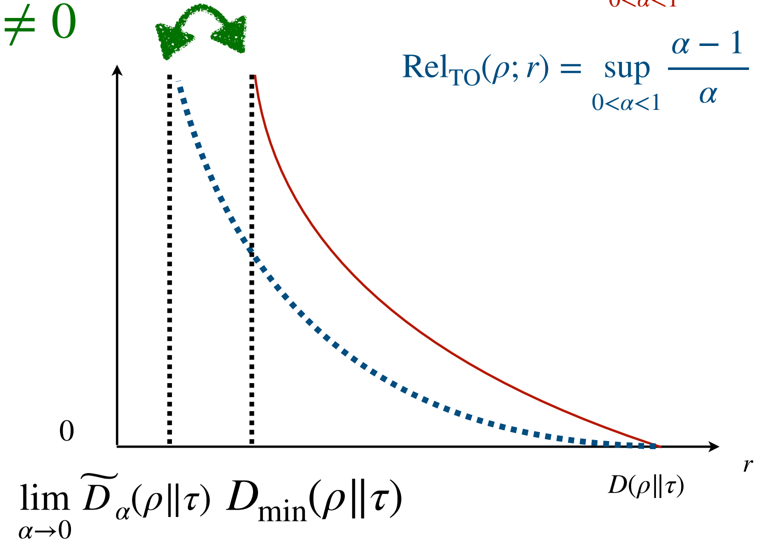
Remark: $[\rho, \tau] = 0 \Rightarrow [\Pi_{\mathrm{supp}(\rho)}, \tau] = 0$

- Example: $[\rho, \tau] \neq 0$ but $[\Pi_{\mathrm{supp}(\rho)}, \tau] = 0$

$$\rho = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} & 0 \\ \frac{1}{4} & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \tau = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{6} \end{pmatrix}$$

$$\mathrm{Rel}_{\mathrm{GPO}}(\rho; r) = \sup_{0<\alpha<1} \frac{\alpha-1}{\alpha}\left(r - \overline{D}_\alpha(\rho\|\tau)\right)$$

$$\mathrm{Rel}_{\mathrm{TO}}(\rho; r) = \sup_{0<\alpha<1} \frac{\alpha-1}{\alpha}\left(r - \widetilde{D}_\alpha(\rho\|\tau)\right)$$



Summary

- **Asymptotic reliability of work extraction** exhibits a sharp operational separation between the axiomatic class (**Gibbs-preserving operations**) and the operational class (**thermal operations**).
- The separation is represented as the difference of divergences. (Petz and sandwiched)
- Our result reveals a **new operation interpletation** of the sandwiched Rényi divergence of $\alpha \in (0,1)$ and the star divergence, which is due to **a symmetry constraint.**
- We show the data-processing inequality of the sandwiched Rényi divergence and the star divergence under a certain covariance, including **time-translation covariance.**

(**Not mentioned in this talk**: we also discussed the bounds of the exponents of general resource distillation/dilution)

Thank you!

Proof

Proposition

Let σ be a quantum state. For a channel Λ satisfying $\mathscr{P}_{\Lambda(\sigma)^{\otimes n}} \circ \Lambda^{\otimes n} \circ \mathscr{P}_{\sigma^{\otimes n}} = \mathscr{P}_{\Lambda(\sigma)^{\otimes n}} \circ \Lambda^{\otimes n}$, we have

$$\widetilde{D}_\alpha(\rho\|\sigma) \geq \widetilde{D}_\alpha\left(\Lambda(\rho)\|\Lambda(\sigma)\right), \quad \forall\alpha \geq 0 \qquad D^\star(\sigma\|\rho) \geq D^\star\left(\Lambda(\sigma)\|\Lambda(\rho)\right)$$

It suffices to discuss the case with $\alpha \in (0,\ 1/2)$. Noting that $\widetilde{D}_\alpha(\rho\|\sigma) = \lim_{n\to\infty} \frac{1}{n} D_\alpha(\mathscr{P}_{\sigma^{\otimes n}}(\rho^{\otimes n})\|\sigma^{\otimes n})$, we have

$$\begin{aligned}
\widetilde{D}_\alpha(\rho\|\sigma) &= \lim_{n\to\infty} \frac{1}{n} D_\alpha(\mathscr{P}_{\sigma^{\otimes n}}(\rho^{\otimes n})\|\sigma^{\otimes n}) = \lim_{n\to\infty} \frac{1}{n} \overline{D}_\alpha(\mathscr{P}_{\sigma^{\otimes n}}(\rho^{\otimes n})\|\sigma^{\otimes n}) \\
&\geq \lim_{n\to\infty} \frac{1}{n} \overline{D}_\alpha(\mathscr{P}_{\Lambda(\sigma)^{\otimes n}} \circ \Lambda^{\otimes n} \circ \mathscr{P}_{\sigma^{\otimes n}}(\rho^{\otimes n})\|\Lambda(\sigma)^{\otimes n}) \\
&= \lim_{n\to\infty} \frac{1}{n} \overline{D}_\alpha(\mathscr{P}_{\Lambda(\sigma)^{\otimes n}} \circ \Lambda^{\otimes n}(\rho^{\otimes n})\|\Lambda(\sigma)^{\otimes n}) \\
&= \lim_{n\to\infty} \frac{1}{n} \widetilde{D}_\alpha(\mathscr{P}_{\Lambda(\sigma)^{\otimes n}} \circ \Lambda(\rho)^{\otimes n}\|\Lambda(\sigma)^{\otimes n}) = \widetilde{D}_\alpha(\Lambda(\rho)\|\Lambda(\sigma))
\end{aligned}$$

DPI of Petz Rényi divergence

Covariance condition

Reliability of work extraction

Theorem : Reliability/ sc exponent of work extraction

- Reliability of asymptotic work extraction with a target work rate r is characterized as

$$\mathrm{Rel}_{\mathrm{GPO}}(\rho; r) = \sup_{0<\alpha<1} \frac{\alpha - 1}{\alpha}\left(r - \overline{D}_\alpha(\rho\|\tau)\right)$$

$$\mathrm{Rel}_{\mathrm{TO}}(\rho; r) = \sup_{0<\alpha<1} \frac{\alpha - 1}{\alpha}\left(r - \widetilde{D}_\alpha(\rho\|\tau)\right)$$

$\mathrm{Rel}_{\mathrm{GPO}}(\rho; r) > \mathrm{Rel}_{\mathrm{TO}}(\rho; r)$ holds for any $\forall r \in (0,\, D(\rho\|\tau))$ whenever $[\rho, H] \neq 0$.

[Hiai, Mosonyi, Ogawa, JMP (2008)]

- The strong converse exponent of work extraction with a target work rate r is characterized as

$$\mathrm{Rel}^{\dagger}_{\mathrm{GPO}}(\rho; r) = \mathrm{Rel}^{\dagger}_{\mathrm{TO}}(\rho; r) = \sup_{\alpha>1} \frac{\alpha - 1}{\alpha}\left(r - \widetilde{D}_\alpha(\rho\|\tau)\right)$$

[Mosonyi, Ogawa, IEEE TI (2015)]

- The separation in reliability is sharply represented as the difference of divergences ($\overline{D}_\alpha(\rho\|\tau) \geq \widetilde{D}_\alpha(\rho\|\tau)$) (strict separation comes from the equality condition of ALT inequality)

- Operational interpretation of $\widetilde{D}_\alpha$ of order $\alpha \in (0,1)$, which arises from **time-translation covariance**