

Partial Trace Relations Beyond Normal Matrices

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- 1 Introduction
- 2 Dilations of Partial Traces
- 3 Partial Trace Inequalities
- 4 Applications

1 Introduction

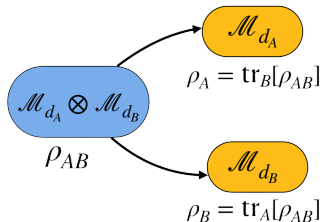
2 Dilations of Partial Traces

3 Partial Trace Inequalities

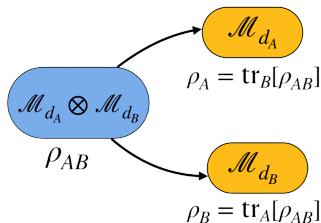
4 Applications

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- We denote $\mathcal{M}_{d_A \otimes d_B} := \mathcal{M}_{d_A} \otimes \mathcal{M}_{d_B}$.

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Notation (Partial Traces)

Given a matrix C on a composite system $\mathcal{M}_{d_A \otimes d_B}$, we denote

$$C_A = \text{tr}_B[C] \in \mathcal{M}_{d_A}, \quad C_B = \text{tr}_A[C] \in \mathcal{M}_{d_B}.$$

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Questions:

- What can the partial trace of a general normal matrix be?
- What can the partial trace of a general non-normal matrix be?
- What about rank-1 general matrices?
- What about rank-2 matrices?

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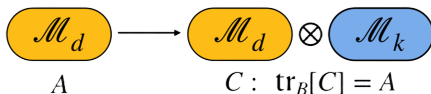
Definition (Dilation)

Given $A \in \mathcal{M}_d$, we say that $C \in \mathcal{M}_{d \otimes k}$ is a dilation of A if $\text{tr}_B[C] = A$.



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Proposition (Structured dilations)

For every $A \in \mathcal{M}_d$ we can find a dilation C (under specific assumptions on k) such that:

- (i) C is a normal matrix ($k = 2$).
- (ii) C is a unitary matrix ($k \geq \|A\|_\infty$).
- (iii) C is a nilpotent matrix ($k = 2$). The condition $\text{tr}[A] = 0$ is necessary.
- (iv) C is an idempotent matrix ($k = \text{tr}[A] \text{rk}[A]$). The conditions $A = 0$ or $\text{tr} A \in \mathbb{N}$ are necessary.

Definition

We say that two matrices $A \in \mathcal{M}_{d_A}$, $B \in \mathcal{M}_{d_B}$ are Flanders-similar if

- 1 They have the same Jordan block structure corresponding to the non-zero eigenvalues.
- 2 Let $a_1 \geq a_2 \geq \dots$ be the sizes of the Jordan blocks of A corresponding to the eigenvalue 0, and let $b_1 \geq b_2 \geq \dots$ be the sizes of the Jordan blocks of B corresponding to the eigenvalue 0. If the shorter sequence is padded with zeros, then $|a_i - b_i| \leq 1$ for all i .

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$$J_1 = \left(\begin{array}{cc|cc|c|c} 0 & 1 & & & & \\ & 0 & & & & \\ \hline & & 0 & 1 & & \\ & & & 0 & & \\ & & & & 0 & \\ & & & & & 0 \end{array} \right) \quad J_2 = \left(\begin{array}{ccc|cc} 0 & 1 & & & \\ & 0 & 1 & & \\ & & 0 & & \\ \hline & & & 0 & 1 \\ & & & & 0 \end{array} \right) \quad \begin{cases} a = (2, 2, 1, 1) \\ b = (3, 2, 0, 0) \end{cases}$$

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Theorem (Simultaneous rank-one dilation)

- If $C \in \mathcal{M}_{d_A \otimes d_B}$ is rank 1, then $B = \text{tr}_A C$ and $A = \text{tr}_B C$ are Flanders-similar.
- Conversely, if two matrices A, B are Flanders-similar, then there is a rank 1 joint dilation C such that $\text{tr}_A C = B$ and $\text{tr}_B C = A$.

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Proposition (Simultaneous rank-two dilation)

For any two matrices $A, B \in \mathcal{M}_d$ there exists a rank-two matrix $C \in \mathcal{M}_{d \otimes d}$ s.t. $\text{tr}_A[C] = B$ and $\text{tr}_B[C] = A \Leftrightarrow \text{tr}[A] = \text{tr}[B]$.

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Theorem (Norm inequality template)

Let $C \in \mathcal{M}_{d_1 \otimes \dots \otimes d_n}$ have rank r and partial traces $C_i \in \mathcal{M}_{d_i}$. For any $\gamma \geq 0$,

$$\sum_{i=1}^n \| C_i \| \leq \gamma \| C \|_1 + \kappa(\gamma) \| C \|, \quad (3)$$

$$\kappa(\gamma) := \sup \phi^*(\Lambda_1, \dots, \Lambda_r), \quad (4)$$

where $\Lambda_1, \dots, \Lambda_r \geq 0$ are obtained from n -vectors $(\lambda^{(i)} \in \mathbb{R}_+^{d_i})_{i=1}^n$ satisfying $\phi^*(\lambda^{(i)}) \leq 1$.

Proposition (Extended Audenaert inequality)

If $C \in \mathcal{M}_{d_1 \otimes \dots \otimes d_n}$ has partial traces $(C_i \in \mathcal{M}_{d_i})_{i=1}^n$, then for all $p \in [1, \infty]$:

$$\sum_{i=1}^n \|C_i\|_p \leq \underbrace{(n-1)}_{\gamma} \|C\|_1 + \underbrace{1}_{\kappa(\gamma)} \|C\|_p. \quad (5)$$

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which for $f(x_1, x_2) = x_1^{\alpha} + x_2^{\alpha}$

$$\Rightarrow \|C_1\|_p^{\alpha} + \|C_2\|_p^{\alpha} \leq \|C\|_1^{\alpha} + \|C\|_p^{\alpha}, \quad \alpha \geq 1. \quad (7)$$

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Werner states are one-parameter family of density operators $\rho_\alpha \in \mathcal{M}_{d \otimes d}$,

$$\rho_\alpha := \frac{\mathbb{1} + \alpha \mathbb{F}}{d^2 + \alpha d}, \quad \alpha \in [-1, 1]. \quad (8)$$

Open Question: Is NPT equivalent to distillability?

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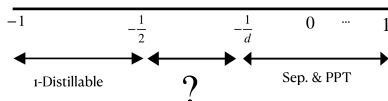
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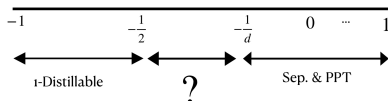
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Theorem⁶

An NPT Werner state ρ_α is 2-copy undistillable \Leftrightarrow for every rank-2 matrix $C \in \mathcal{M}_{d \otimes d}$:

$$\|C_A\|_2^2 + \|C_B\|_2^2 \leq |\alpha| |\operatorname{tr}[C]|^2 + \frac{1}{|\alpha|} \|C\|_2^2. \quad (9)$$

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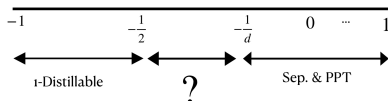
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Distillability of Werner states

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Proposition

Werner states are 2-copy undistillable for $\alpha \geq -1/3$.

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$$\mathcal{H}_A \quad \mathcal{H}_B$$

$$\left(\mathbb{C}^d \otimes \dots \otimes \mathbb{C}^d \right) \otimes \left(\mathbb{C}^d \otimes \dots \otimes \mathbb{C}^d \right)$$

The diagram shows two identical tensor product structures, \mathcal{H}_A and \mathcal{H}_B , each enclosed in an oval. Each structure consists of n copies of \mathbb{C}^d connected by tensor products (\otimes). The two structures are then connected to each other by a larger tensor product symbol (\otimes).

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Let

$$\omega_{(i)} = \mathbb{1}_{A_i B_i} \otimes \left(\bigotimes_{\substack{j=1 \\ j \neq i}}^n |\Omega_j\rangle \langle \Omega_j|_{A_j B_j} \right) \quad (10)$$

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Proposition

For every $0 \leq k < d$,

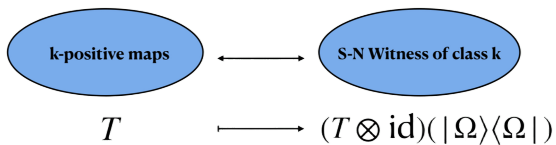
$$W_k = (1 + (n-1)k)\mathbb{1} - \sum_{i=1}^n \omega_{(i)}, \quad (11)$$

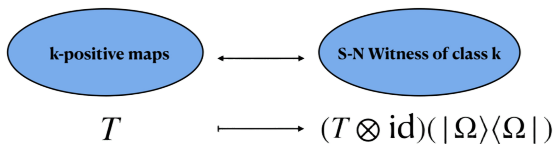
is a Schmidt-number witness of class $k+1$, i.e.

$$\text{tr}[W_k \sigma] \geq 0, \quad \forall \sigma \text{ with } \text{SN}(\sigma) \leq k.$$

and there exists ρ with Schmidt-number $k+1$ such that

$$\text{tr}[W_k \rho] < 0.$$





Corollary

For $k, n \in \mathbb{N}$ the following map on $(\mathcal{M}_d)^{\otimes n}$ is k -positive but not $(k+1)$ -positive:

$$T(X) := (1 + (n-1)k) \text{tr}[X] \mathbb{1} - \sum_{i=1}^n \text{tr}_i[X] \otimes \mathbb{1}_{\{1, \dots, n\} \setminus \{i\}} \quad (12)$$

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