Power and limitations of distributed state purification

Benchi Zhao, Yu-Ao Chen, Xuanqiang Zhao, Chengkai Zhu,
Giulio Chiribella and Xin Wang

PRL 136,090203

Date: 2026.06.25

香 港 大 學
THE UNIVERSITY OF HONG KONG

Quantum Information and Computation Initiative

Contents

Background & Motivation

- Quantum computing provide exponential speed up in certain problems
- Two main obstacles: **Scalability + Noise**

Background & Motivation

- Quantum computing provide exponential speed up in certain problems
- Two main obstacles: **Scalability** + **Noise**

Current Quantum Device

~1000 qubits

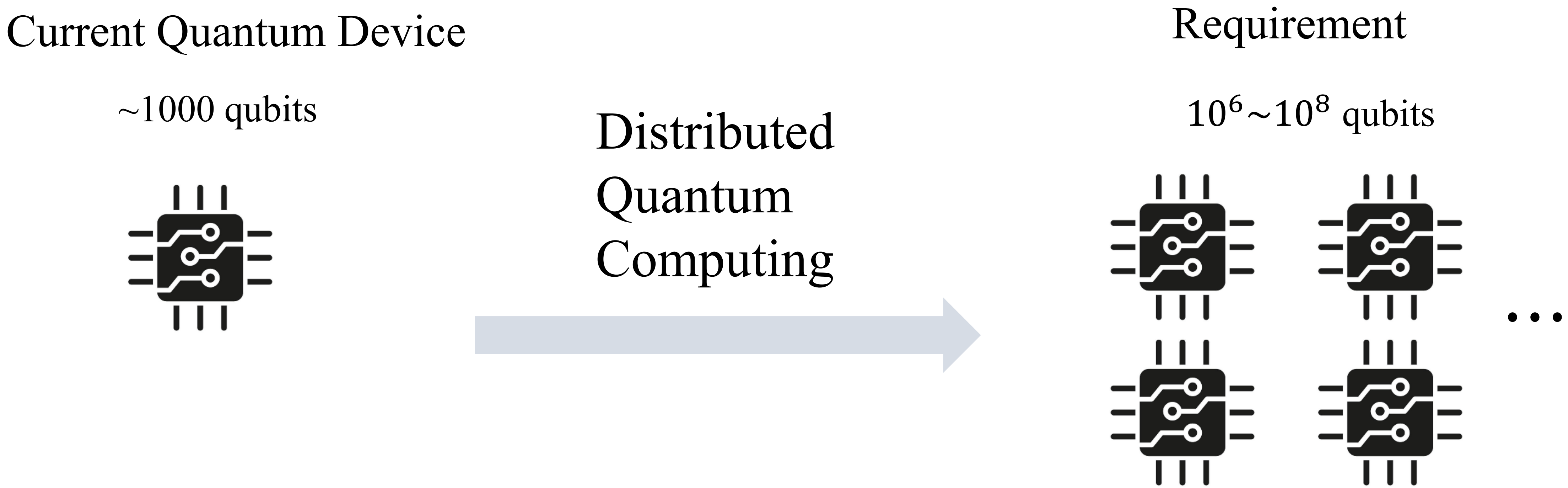
?

Requirement

$10^6 \sim 10^8$ qubits

Background & Motivation

- Quantum computing provide exponential speed up in certain problems
- Two main obstacles: **Scalability** + **Noise**

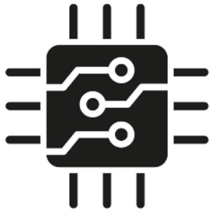


Background & Motivation

- Quantum computing provide exponential speed up in certain problems
- Two main obstacles: **Scalability** + **Noise**

Current Quantum Device

~1000 qubits

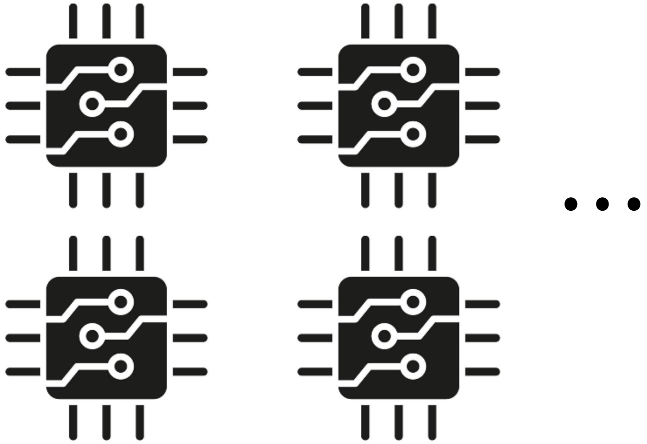


Distributed Quantum Computing

WHAT ABOUT NOISE?

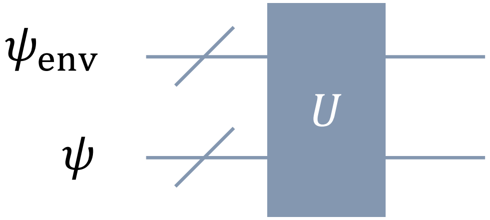
Requirement

$10^6 \sim 10^8$ qubits



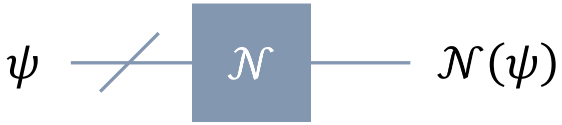
Background & Motivation

- Quantum systems usually suffer from unwanted interactions with the environment



$\psi = |\psi\rangle\langle\psi|$ is a pure state

- Noises can be modeled as **quantum channels** $\mathcal{N}$ (completely positive and trace-preserving (CPTP) maps)



For example, a depolarizing channel is a common noise model in quantum computing which maps quantum state into identity probabilistically

$$\mathcal{N}(\psi) = (1 - p)\psi + p\frac{I}{2}$$

where $0 \leq p \leq 1$

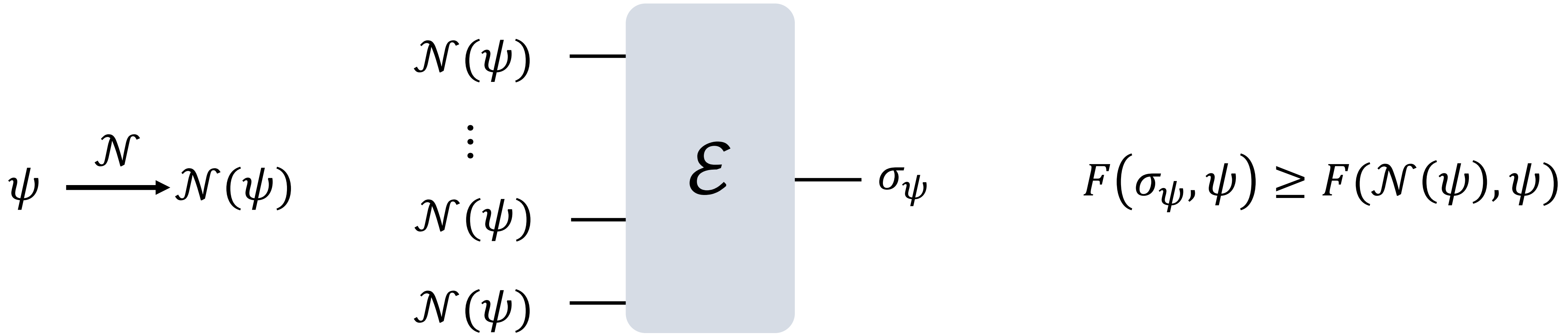
Definitions

Quantum state purification is a promising method to reduce quantum noise

Suppose we have n-copy noise states $\mathcal{N}(\psi)$, where ψ is the pure state and $\mathcal{N}$ is the noise channel. Let $\mathcal{E}$ be an $n \to 1$ completely positive and trace-non-increasing (CPTN) map. Then the output state is

$$\sigma_\psi = \frac{\tilde{\sigma}_\psi}{p_\psi} = \frac{\mathcal{E}(\mathcal{N}(\psi)^{\otimes n})}{Tr[\mathcal{E}(\mathcal{N}(\psi)^{\otimes n})]}$$

If the output state σ_ψ is closer to the target state ψ, i.e., $F(\sigma_\psi, \psi) \geq F(\mathcal{N}(\psi), \psi)$, then we say the CPTN map $\mathcal{E}$ is a ***purification protocol*** for a state ψ.



Definitions

Quantum state purification is a promising method to reduce quantum noise

If the state ψ is sampled from a set of pure states $\mathcal{S}$, and if the map $\mathcal{E}$ can increase the fidelity for **every state ψ in the set** $\mathcal{S}$, i.e.,

$$F(\sigma_\psi, \psi) \geq F(\mathcal{N}(\psi), \psi), \forall \psi \in \mathcal{S}$$

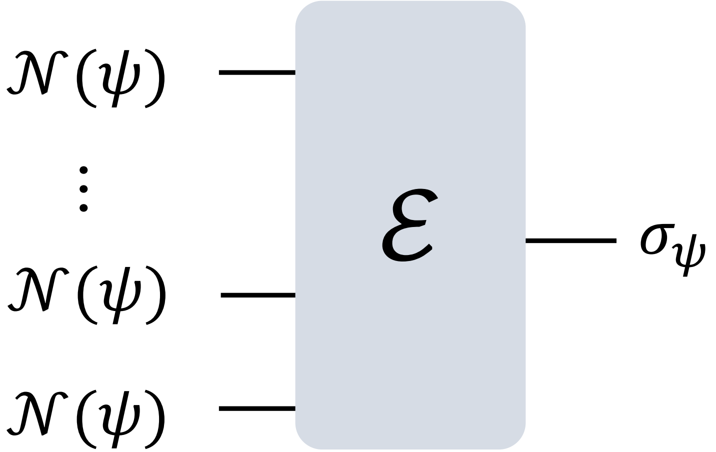
Then we say $\mathcal{E}$ is a ***purification protocol*** for the set $\mathcal{S}$

If the operation $\mathcal{E}$ is a purification protocol for the set $\mathcal{S}$, which contains all **pure states**, then $\mathcal{E}$ is called ***universal purification protocol***.

$$\mathcal{S} = \{\phi_1, \cdots, \phi_n\}$$

$$\downarrow \mathcal{N}$$

$$\mathcal{S} = \{\mathcal{N}(\phi_1), \cdots, \mathcal{N}(\phi_n)\}$$

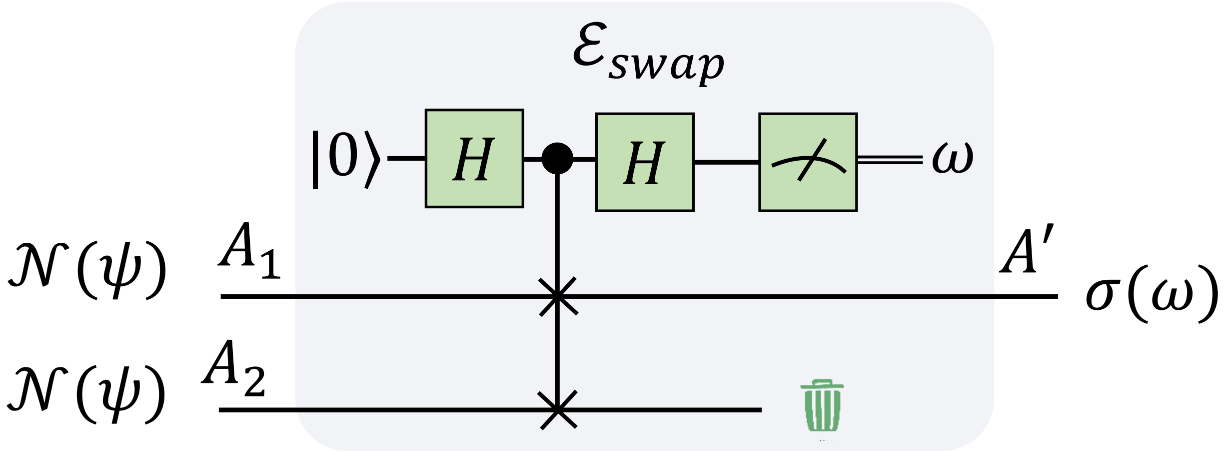


$$F(\sigma_\psi, \psi) \geq F(\mathcal{N}(\psi), \psi), \forall \psi \in \mathcal{S}$$

Preliminaries

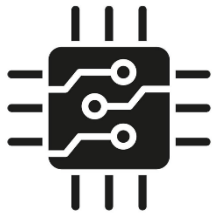
Example of universal purification protocol

$\mathcal{N}(\psi) = \lambda_0\psi + \sum_{i=1}\lambda_1\psi_i$, with $\lambda_0 \geq \lambda_1 \geq \cdots \geq \lambda_n$



- Apply the circuit to noisy states.
- Make measurement on the ancilla qubit.
- If the measurement $\omega = 0$, then the purification process is successful, and output the purified state $\sigma(0)$; Otherwise if the measurement is $\omega = 1$, then the purification is failed, discard the state $\sigma(1)$.
- The successfully purified state is

$$\sigma(0) = \frac{\mathcal{N}(\psi) + \mathcal{N}(\psi)^2}{Tr[\mathcal{N}(\psi) + \mathcal{N}(\psi)^2]}$$



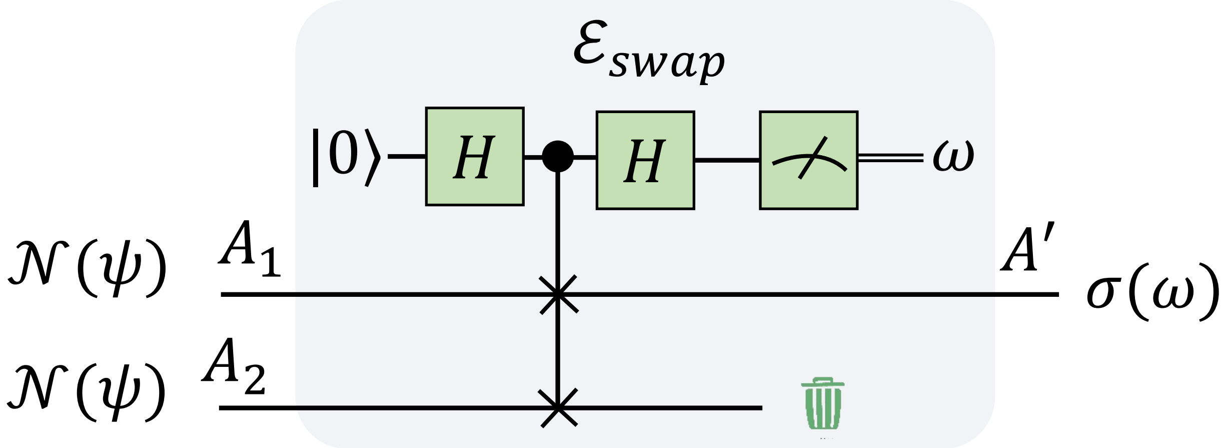
Single Chip

$Tr[\sigma(0) \cdot \psi] > Tr[\mathcal{N}(\psi) \cdot \psi], \forall \psi$

Preliminaries

Example of universal purification protocol

$\mathcal{N}(\psi) = \lambda_0\psi + \sum_{i=1}\lambda_1\psi_i$, with $\lambda_0 \geq \lambda_1 \geq \cdots \geq \lambda_n$



- Apply the circuit to noisy states.
- Make measurement on the ancilla qubit.
- If the measurement $\omega = 0$, then the purification process is successful, and output the purified state $\sigma(0)$; Otherwise if the measurement is $\omega = 1$, then the purification is failed, discard the state $\sigma(1)$.
- The successfully purified state is

$$\sigma(0) = \frac{\mathcal{N}(\psi) + \mathcal{N}(\psi)^2}{Tr[\mathcal{N}(\psi) + \mathcal{N}(\psi)^2]}$$

Is there a similar protocol for distributed computing?

$$Tr[\sigma(0) \cdot \psi] > Tr[\mathcal{N}(\psi) \cdot \psi], \forall\psi$$

Definitions

The purification protocol is not unique. Which one has the best performance?

Average Fidelity: For a given pure state set $\mathcal{S}$, noise channel $\mathcal{N}$ and the $n \rightarrow 1$ distributed purification protocol $\mathcal{E}$ with average success probability $\bar{P}$, the average purification fidelity is defined as

$$\bar{F}(n, \bar{P}; \mathcal{S}, \mathcal{N}, \mathcal{E}) = \frac{1}{|\mathcal{S}|}\sum_{\psi\in\mathcal{S}} F\left(\psi, \frac{\tilde{\sigma}_\psi}{\bar{P}}\right)$$

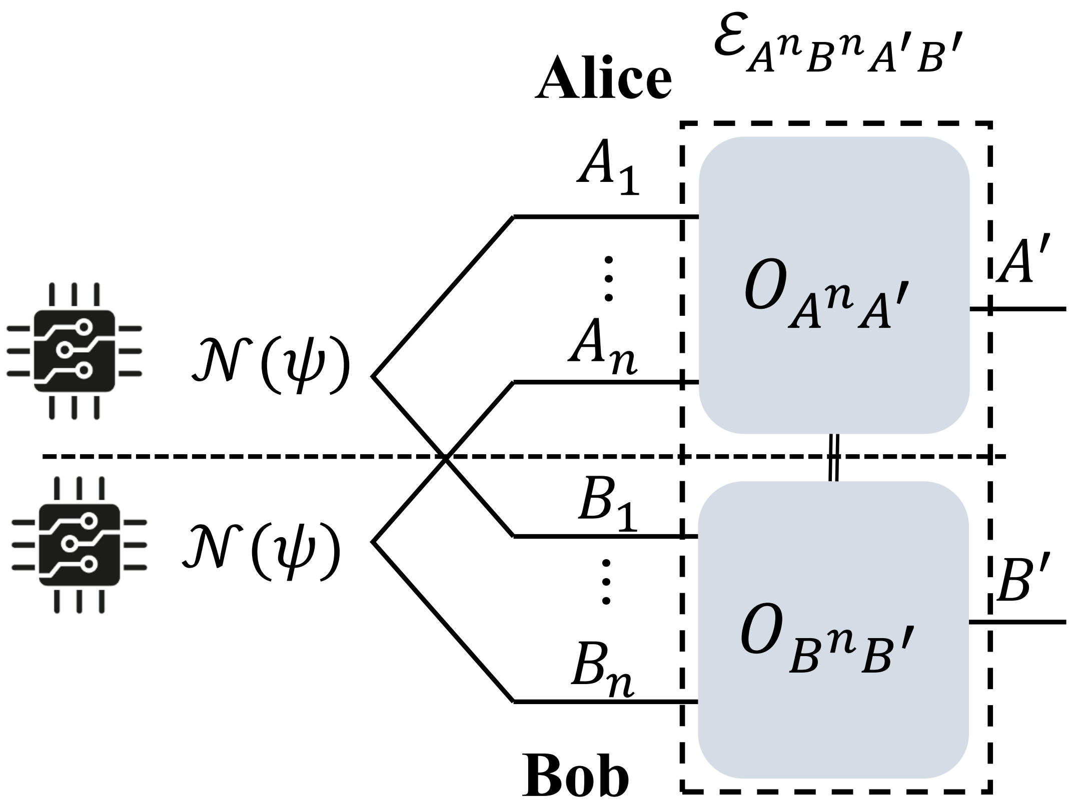
where $\tilde{\sigma}_\psi$ is the unnormalized purified state.

The **optimal purification protocol** $\mathcal{E}^*$ is defined by the one achieves the optimal average fidelity.

$$\bar{F}^*(n, \bar{P}; \mathcal{S}, \mathcal{N}) = \max_{\mathcal{E}}\{\bar{F}\,(n, \bar{P}; \mathcal{S}, \mathcal{N}, \mathcal{E}) | \, \mathcal{E} \in CPTN\,, \frac{1}{|S|}\sum_{\psi\in\mathcal{S}} Tr\left[\mathcal{E}\left(\mathcal{N}(\psi)^{\otimes n}\right)\right] = \bar{P}\}$$

Task

Distributed quantum computing enables spatially separated quantum processors to perform joint computations via **local operations and classical communication (LOCC).**



For a quantum noise $\mathcal{N}$ and a set of pure state $\mathcal{S}$, whether there exist an $n \to 1$ LOCC-CPTN map $\mathcal{E}_{A^nB^nA'B'}$, such that

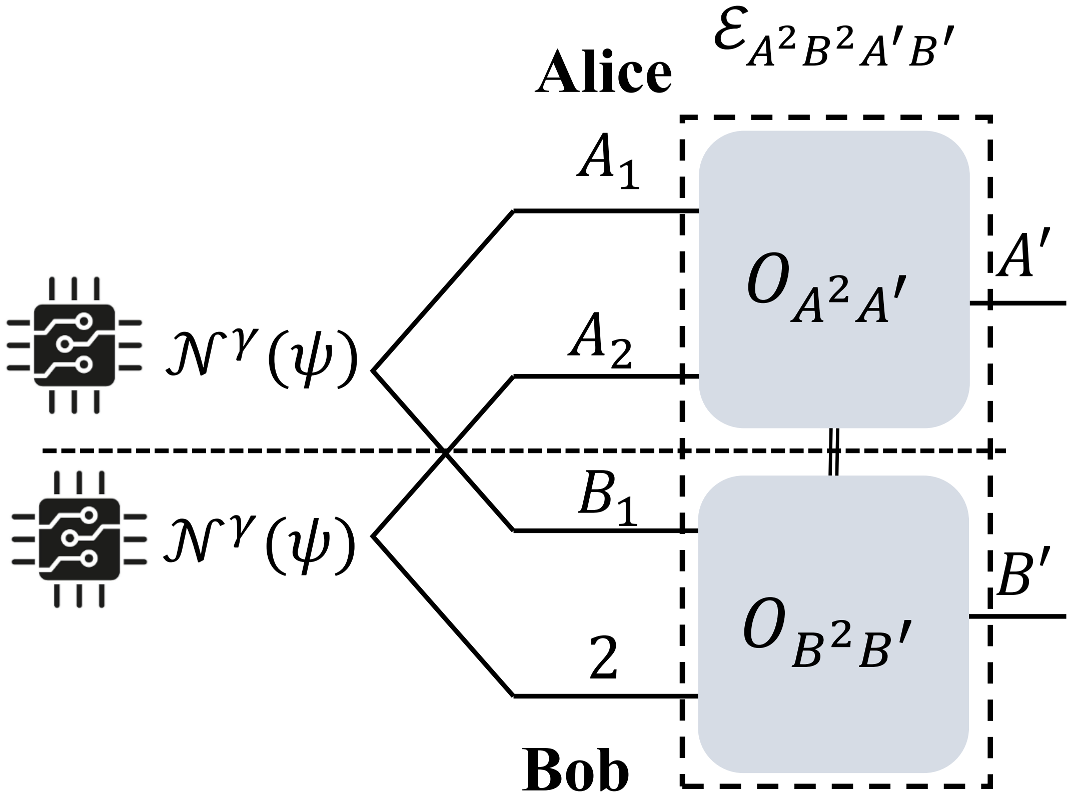
$$\frac{Tr[\mathcal{E}_{A^nB^nA'B'}(\mathcal{N}(\psi)^{\otimes n}) \cdot \psi]}{Tr[\mathcal{E}_{A^nB^nA'B'}(\mathcal{N}(\psi)^{\otimes n})]} \geq Tr[\mathcal{N}(\psi) \cdot \psi], \forall \psi \in \mathcal{S}$$

whether there exist a map $\mathcal{E}_{A^nB^nA'B'}$ such that

$$\frac{Tr[\mathcal{E}_{A^nB^nA'B'}(\mathcal{N}(\psi)^{\otimes n}) \cdot \psi]}{Tr[\mathcal{E}_{A^nB^nA'B'}(\mathcal{N}(\psi)^{\otimes n})]} = Tr[\mathcal{N}(\psi) \cdot \psi], \forall \psi \in \mathcal{S}$$

Then we say the purification protocol is **trivial**

No-go theorem



Let's consider the following setting:

# of Copy:

$$n = 2$$

Quantum Noise: Depolarizing noise $\mathcal{N}^\gamma$

$$\mathcal{N}^\gamma(\psi) = (1-\gamma)\psi + \gamma\frac{I}{d}$$

Quantum state set $\mathcal{S}_P$

$$\mathcal{S}_P = \{\psi\}, \forall\psi = |\psi\rangle\langle\psi|$$

Theorem 1

For the depolarizing noise channel $\mathcal{N}^\gamma$ with noise level γ, there is no nontrivial $2 \rightarrow 1$ LOCC purification protocol $\mathcal{E}_{A^nB^nA'B'}$ for the set that contains all pure states $\mathcal{S}_P$

No-go theorem

Theorem 1

For the depolarizing noise channel $\mathcal{N}^{\gamma}$ with noise level γ, there is no nontrivial $2 \rightarrow 1$ LOCC purification protocol $\mathcal{E}_{A^n B^n A' B'}$ for the set that contains all pure states $\mathcal{S}_P$.

Prove Sketch:

⇒ Relax LOCC to positive partial transpose (PPT) operations.

⇒ Translate this problem into an SDP.

⇒ Simplify with the SDP with the symmetry properties.

⇒ Prove that the optimal solution to the SDP is the trivial purification protocol, implying there is no non-trivial distributed purification protocol.

No-go theorem

Let's consider a simpler case:

# of Copy:

$$n = 2$$

Quantum Noise: Depolarizing noise $\mathcal{N}^{\gamma}$

$$\mathcal{N}^{\gamma}(\psi) = (1-\gamma)\psi + \gamma\frac{I}{d}$$

Quantum state set $\mathcal{S}_B$, $\mathcal{S}_{MES}$

$$\mathcal{S}_B = \{\Phi^{\pm}, \Psi^{\pm}\}$$

$$\mathcal{S}_{MES} = \{\psi | \psi = U \otimes V\Phi^{+}U^{\dagger} \otimes V^{\dagger}, \forall U, V \in SU(2)\}$$

No-go theorem

Let's consider a simpler case:

# of Copy:

$$n = 2$$

Quantum Noise: Depolarizing noise $\mathcal{N}^{\gamma}$

$$\mathcal{N}^{\gamma}(\psi) = (1-\gamma)\psi + \gamma\frac{I}{d}$$

Quantum state set $\mathcal{S}_B$, $\mathcal{S}_{MES}$

$$\mathcal{S}_B = \{\Phi^{\pm}, \Psi^{\pm}\}$$

$$\mathcal{S}_{MES} = \{\psi | \psi = U \otimes V\Phi^{+}U^{\dagger} \otimes V^{\dagger}, \forall U, V \in SU(2)\}$$

Theorem 2

For the depolarizing noise channel $\mathcal{N}^{\gamma}$ with noise level γ, there is no nontrivial $2 \rightarrow 1$ LOCC purification protocol $\mathcal{E}_{A^n B^n A' B'}$ for the set that contains all maximally entangled states $\mathcal{S}_{MES}$

Theorem 3

For the depolarizing noise channel $\mathcal{N}^{\gamma}$ with noise level γ, there is no nontrivial $2 \rightarrow 1$ LOCC purification protocol $\mathcal{E}_{A^n B^n A' B'}$ for the set that contains 4 Bell states $\mathcal{S}_B$

Go theorem

The $2 \rightarrow 1$ no-go theorem does not imply the $n \rightarrow 1$ no-go theorem

- If Alice and Bob share infinite many copies of state, then they could process tomography first to get the density matrix of the noisy state $\mathcal{N}(\psi)$.
- Since the quantum noise is known, they can derive what state ψ should be
- Now, the problem reduce to

For a quantum noise $\mathcal{N}$ and a pure state ψ, if there exist an $n \rightarrow 1$ LOCC-CPTN map $\mathcal{E}_{A^n B^n A' B'}$, such that

$$\frac{Tr[\mathcal{E}_{A^n B^n A' B'}\left(\mathcal{N}(\psi)^{\otimes n}\right) \cdot \psi]}{Tr[\mathcal{E}_{A^n B^n A' B'}(\mathcal{N}(\psi)^{\otimes n})]} > Tr[\mathcal{N}(\psi) \cdot \psi]$$

Go theorem

Let's consider the following setting:

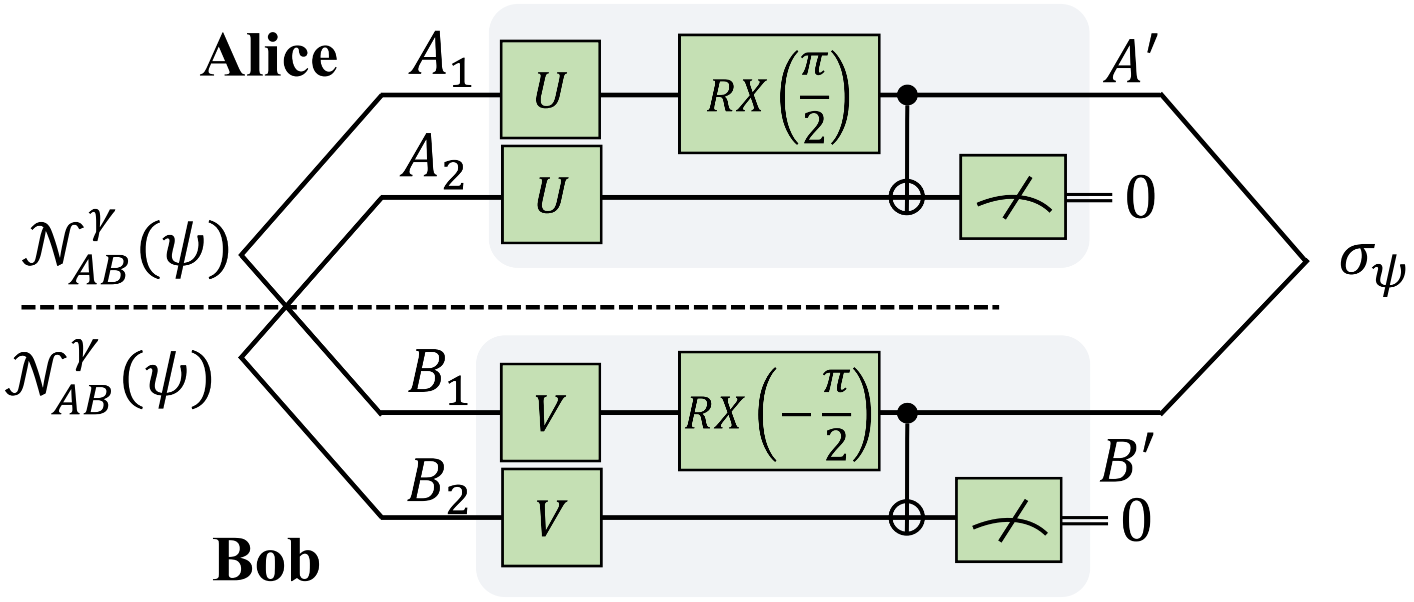
# of Copy: $n = 2$

Quantum Noise: Depolarizing noise

$$\mathcal{N}^{\gamma}(\psi) = (1-\gamma)\psi + \gamma\frac{I}{d}$$

Theorem 4

For the depolarizing noise channel $\mathcal{N}^{\gamma}$ with noise level γ ∈ [0, 0.4],there always exist a nontrivial LOCC purification protocol for a given state ψ.



$$U \otimes V(\psi) = \alpha|00\rangle + \beta|11\rangle$$

With calculations, we can find

$$Tr[\sigma_\psi \cdot \psi] > Tr[\mathcal{N}_{AB}^{\gamma}(\psi) \cdot \psi]$$

Purification protocol with optimization

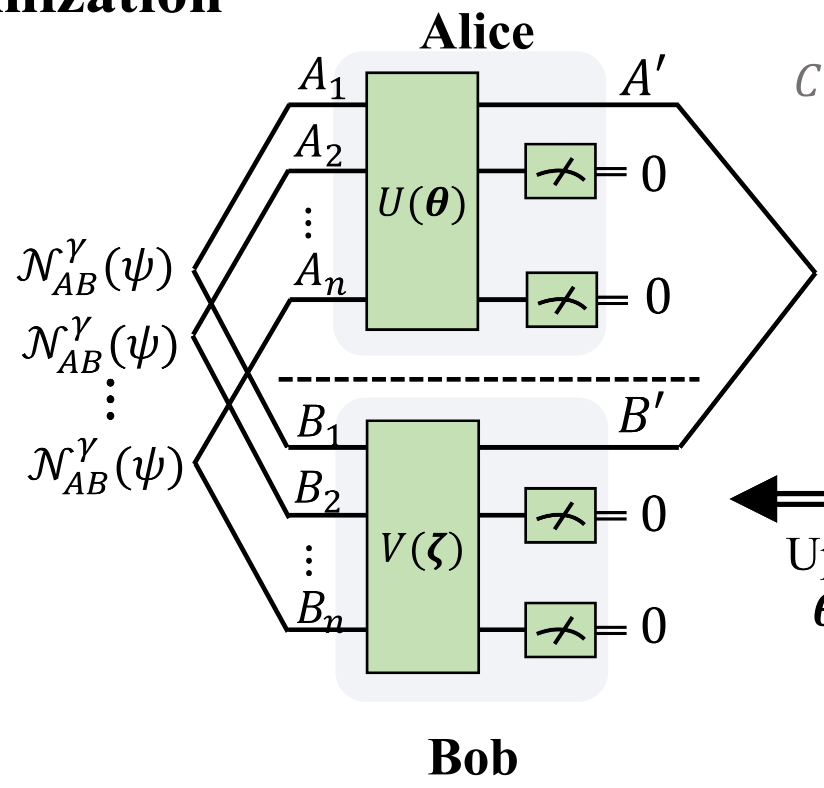
In previous protocol, it can only purify a single pure state. If there exist a purification protocol that can purify a set with multi-state set $\mathcal{S}$?

Here we propose a method to design distributed purification protocol via optimization

Purification protocol with optimization

In previous protocol, it can only purify a single pure state. If there exist a purification protocol that can purify a set with multi-state set $\mathcal{S}$?

Here we propose a method to design distributed purification protocol via optimization



$$C(\boldsymbol{\theta},\boldsymbol{\zeta}) = -\sum_{\psi\in\mathcal{S}} F(\psi,\sigma_\psi) + \alpha\sum_{\psi\in\mathcal{S}}\left(S\left(F(\psi,\mathcal{N}(\psi)) - F(\psi,\sigma_\psi)\right)\right)$$

Estimate cost function $C(\boldsymbol{\theta},\boldsymbol{\zeta})$

Optimize parameters $\boldsymbol{\theta},\boldsymbol{\zeta}$ to minimize cost function $C(\boldsymbol{\theta},\boldsymbol{\zeta})$

Update $\boldsymbol{\theta}',\boldsymbol{\zeta}'$

$S(x)$ is the step function
$S(x) = x, if\ x \geq 0$
$S(x) = 0, if\ x < 0$

Purification protocol with optimization

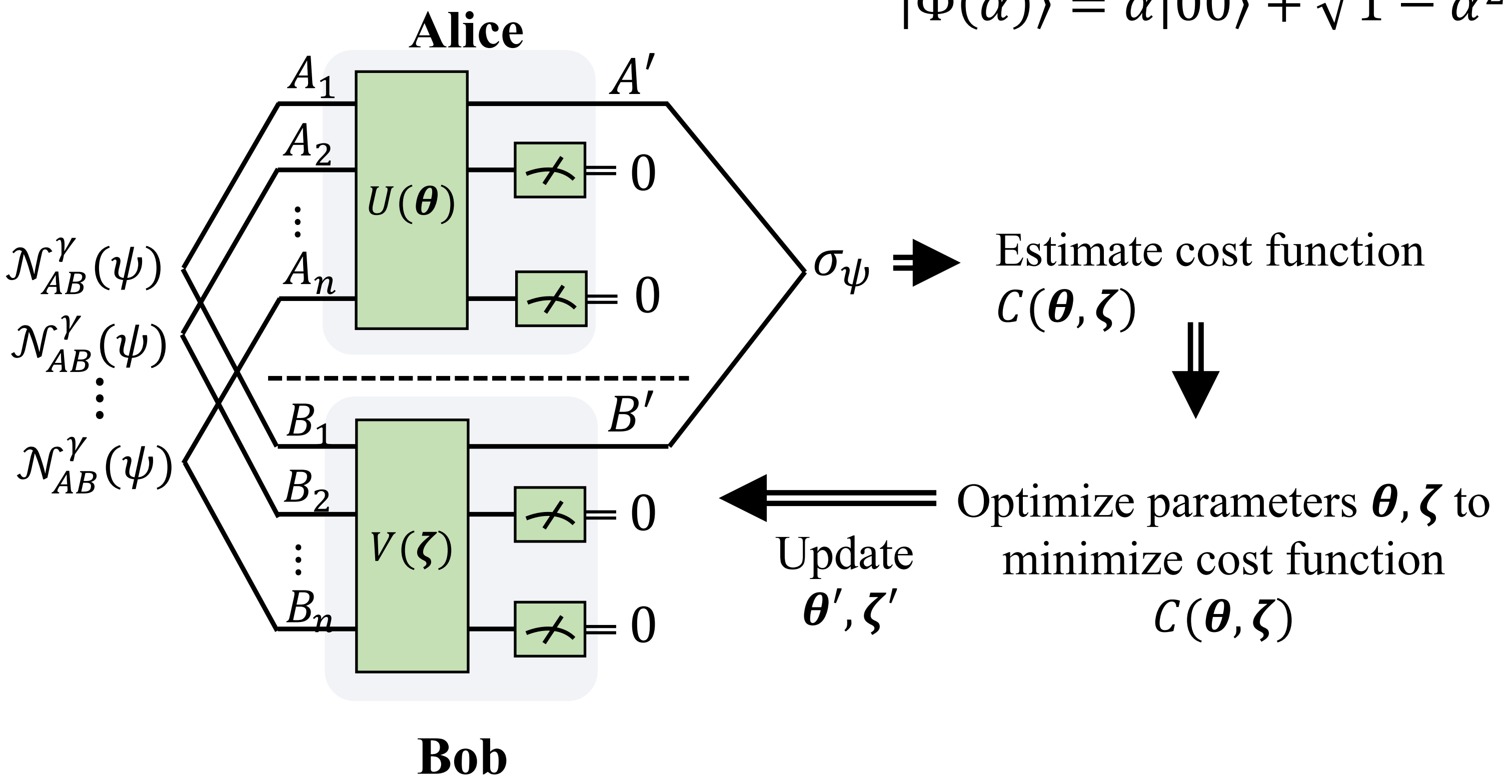
Let's consider the following setting:

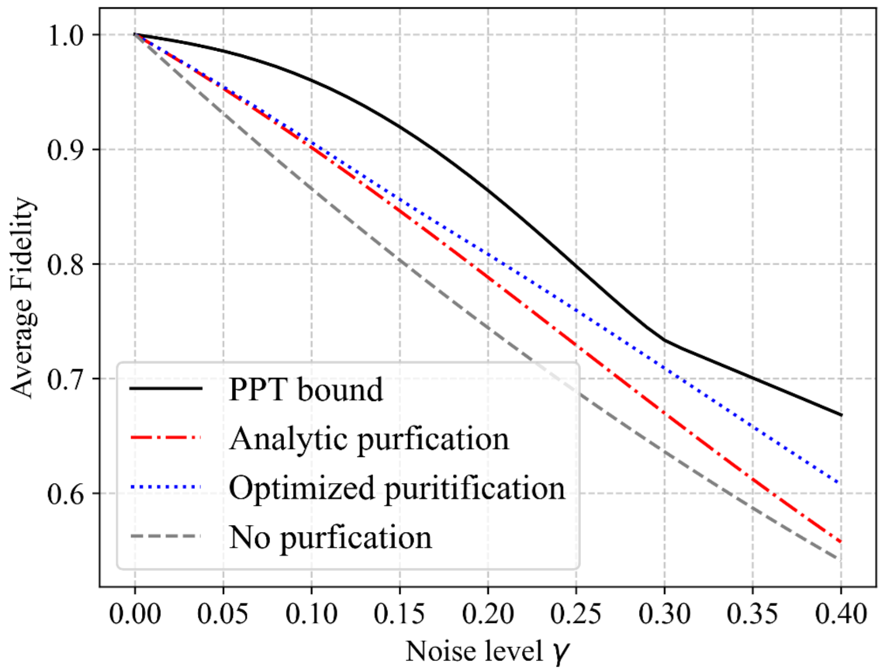
# of Copy: $n = 2$

Quantum Noise: Depolarizing noise: $\mathcal{N}^{\gamma}(\psi) = (1-\gamma)\psi + \gamma\frac{I}{d}$

Quantum state set $\mathcal{S} = \left\{\Phi\left(\frac{1}{\sqrt{2}}\right), \Phi\left(\frac{1}{\sqrt{3}}\right), \Phi\left(\frac{1}{\sqrt{4}}\right), \Phi\left(\frac{1}{\sqrt{5}}\right)\right\}$

$$|\Phi(\alpha)\rangle = \alpha|00\rangle + \sqrt{1-\alpha^2}|11\rangle$$





Conclusion

- For the depolarizing noise $\mathcal{N}^{\gamma}$, there is no non-trivial $2 \rightarrow 1$ LOCC purification protocol for pure state set $\mathcal{S}_P$, maximally entangled state set $\mathcal{S}_{MES}$, and Bell state set $\mathcal{S}_B$.

- For the depolarizing noise $\mathcal{N}^{\gamma}$ and a state ψ, there exist $2 \rightarrow 1$ LOCC purification protocol.

- If the state set is finite, we propose a framework that could help us to design the LOCC purification protocol via optimization.

Thank you for your attention!

Yu-Ao Chen

Xuanqiang Zhao

Chengkai Zhu

Giulio Chiribella

Xin Wang