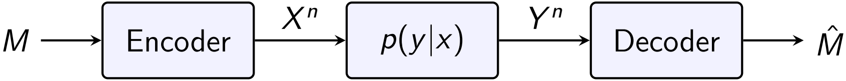
The Auxiliary Receiver Approach in Network Information Theory

A Review

Amin Gohari

June 26, 2026

Capacity of a Classical Point-to-Point Channel



Channel Capacity (Shannon)

$$C = \max_{p(x)} I(X;Y)$$

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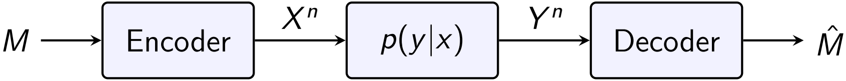

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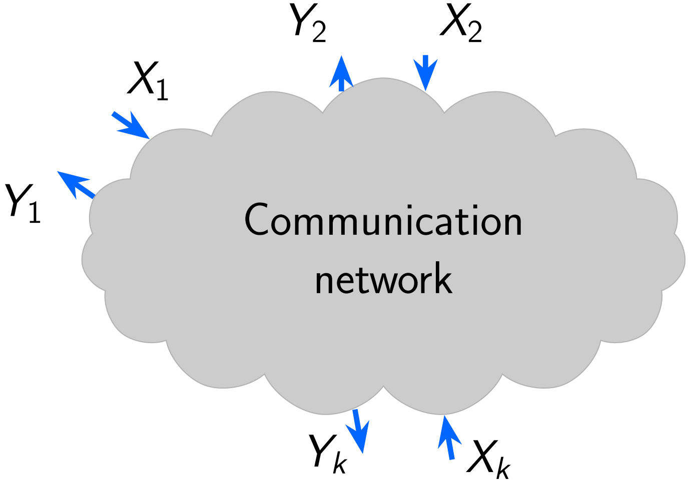
Subadditivity

$$I(X^n;Y^n) \leq \sum_{i=1}^{n} I(X_i;Y_i)$$

We call the LHS the *n-letter expression*, and the above process *single-letterization*.

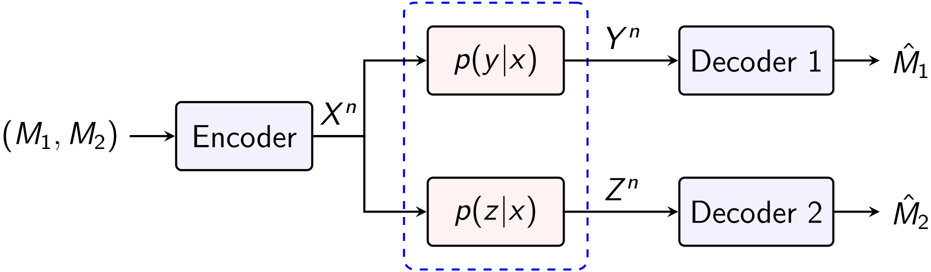
Network Information Theory: Multiple transmitters or receivers



Communication network $p(y_{1:k}|x_{1:k})$

Example: Broadcast Channel

Sending independent messages M_1 and M_2 to two receivers:



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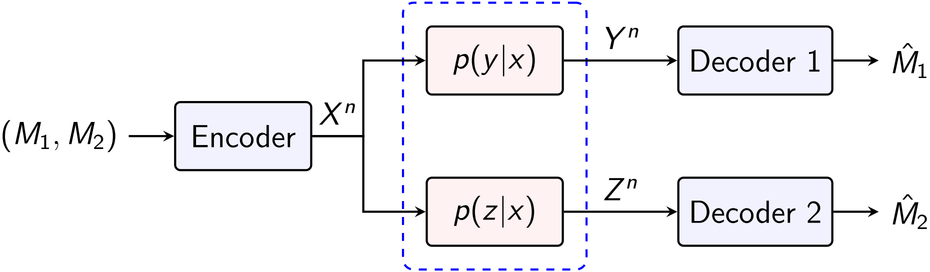

- Tradeoff between message rates R_1 and R_2 $\Rightarrow$ consider the **capacity region**.
- n-letter expressions: $I(M_1; Y^n)$ and $I(M_2; Z^n)$.

Subadditivity for Broadcast Channels: Inner Bound

Operational Capacity Region $\mathcal{C}$

The capacity region $\mathcal{C}$ is the closure of the set of rate pairs (R_1, R_2) such that:

$$R_1 = \frac{1}{n} I(M_1; Y^n), \qquad R_2 = \frac{1}{n} I(M_2; Z^n)$$

taking the union over all blocklengths n and distributions p_{M_1,M_2,X^n}, where M_1 and M_2 are independent and form the Markov chain $(M_1, M_2) \to X^n \to (Y^n, Z^n)$.

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Single-Letter Inner Bound $\mathcal{R}$

Let $\mathcal{R}$ be the set of pairs (R_1, R_2) such that $R_1 = I(U;Y)$ and $R_2 = I(V;Z)$ for some distribution $p_U p_V p_{X|UV}$ where $(U,V) \to X \to (Y,Z)$ forms a Markov chain.

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We have $\mathcal{R} \subset \mathcal{C}$. Take n i.i.d. copies of (U, V, X, Y, Z) and define $M_1 = U^n$ and $M_2 = V^n$.

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We have $\mathcal{R} \subset \mathcal{C}$. Take n i.i.d. copies of (U, V, X, Y, Z) and define $M_1 = U^n$ and $M_2 = V^n$. Here, U and V are called **transmitter-side auxiliary random variables**.

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Open Problem

Finding the exact capacity region for the general broadcast channel remains an **open problem**.

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Techniques for Outer Bounds

- **Standard technique:** Use *transmitter-side* auxiliary random variables that incorporate past/future channel variables.
- **Recent idea:** *Receiver-Side Auxiliary Variables* [This talk!].

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Receiver-side auxiliaries represent *new* / *artificial* outputs that do not exist in the original network.

- They are *silent observers*: no feedback, no encoding, no decoding obligations.
- Why useful?
  - Induce helpful Markov structure.
  - Enable telescoping expansions with past/future of auxiliary outputs.
  - Probe the **surface/curvature** of the capacity region as a function of the channel law.
  - Related: the **copy lemma** augments the probability space with new variables.

Outline

1. Applications of the Auxiliary Receivers
    - Source Coding: Remote Sources
    - Channel Coding: Genie-Auxiliary Receivers
    - Channel Coding: Non-Genie Auxiliary Receiver
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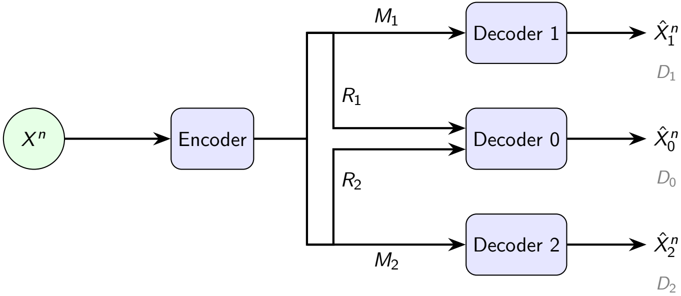
- **Example: Multiple-Description Problem**
  Ozarow (1980): although a remote source plays no direct role in encoding/decoding, it yields a key lower bound.



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- Related works: Reznic–Feder–Zamir (2006), Wang–Viswanath (2007), Wagner–Tavildar–Viswanath (2008), Wang–Viswanath (2009), Chen (2009), Wang–Chen–Wu (2010), Wang–Chen (2013), Song–Shao–Chen (2014), Song–Chen–Tian (2016).
- Similar manipulation in channel coding: Saeedi Bidokhti & Kramer (2016).

Inducing Conditional Independence

Remote sources can induce conditional independence.
Example (Wagner & Anantharam, 2008): choose Z such that

$$p(y_1, \ldots, y_t, z) = p(z) \prod_{i=1}^{t} p(y_i | z).$$

This factorization is useful in multi-terminal source coding.

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Genie-Aided Proofs: Concept & Applications

- Consider communication over a general communication network $p(y_{1:k}|x_{1:k})$ with k nodes.
- Capacity region $\mathsf{C}\big(p(y_{1:k}|x_{1:k})\big)$ as a function of $p(y_{1:k}|x_{1:k})$.

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- There is a class $\mathcal{P}$ of channels for which we know the capacity (e.g., in broadcast channels, less noisy or more capable class).
- We say that $p(z_{1:k}|x_{1:k})$ is an **a Genie-aided** enhancement of $p(y_{1:k}|x_{1:k})$ if

$$Z_i = (Y_i, G_i)$$

for some $p(g_{1:k}|x_{1:k}, y_{1:k})$.

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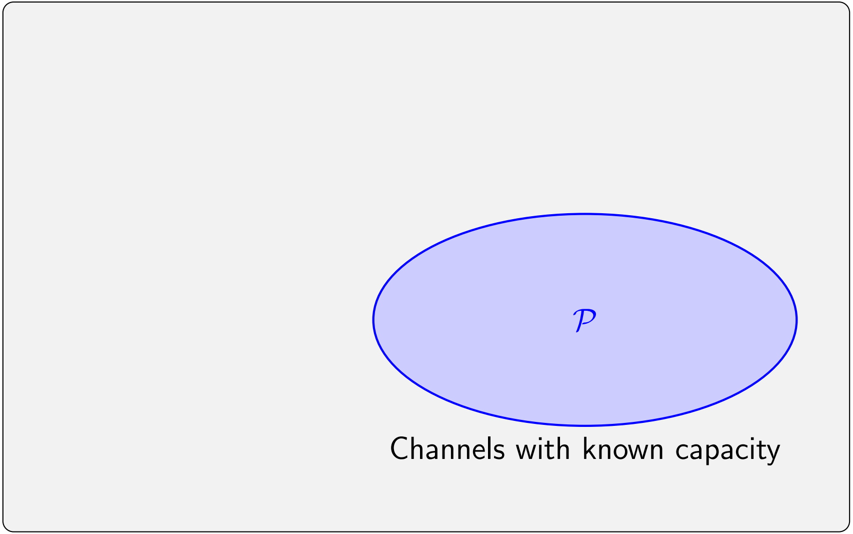
  $$Z_i = (Y_i, G_i)$$

  for some $p(g_{1:k}|x_{1:k}, y_{1:k})$.

  Since extra information cannot hurt,

  $$C(p(y_{1:k}|x_{1:k})) \subseteq C(p(z_{1:k}|x_{1:k})).$$

Space of all Channels $p(\cdot|\cdot)$



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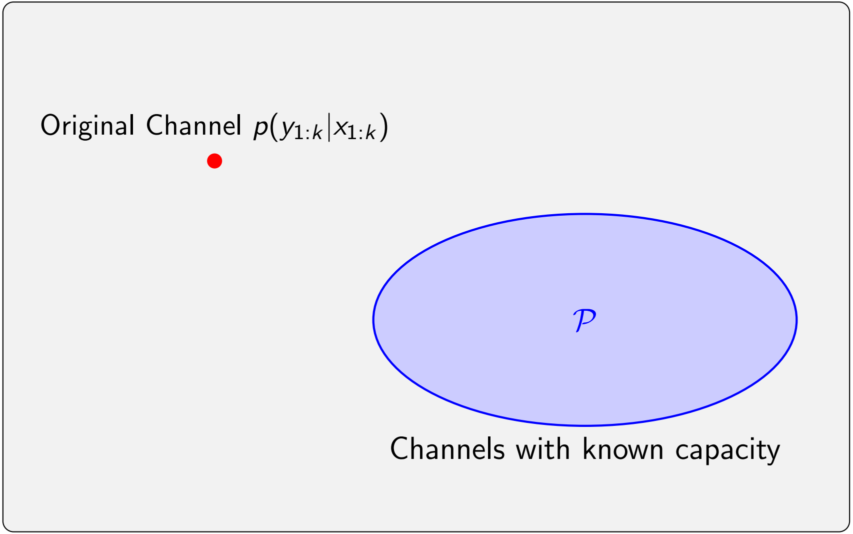
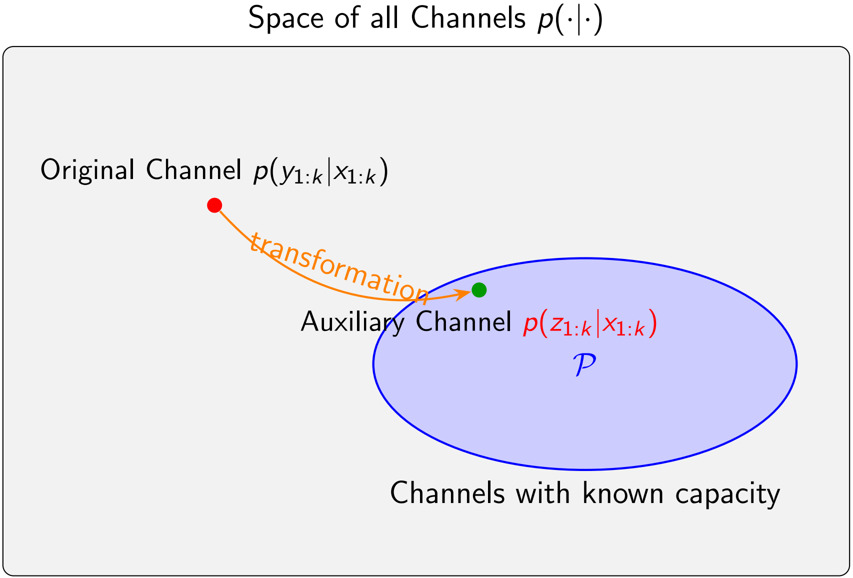


Original Channel $p(y_{1:k}|x_{1:k})$

$\mathcal{P}$

Channels with known capacity



Channel enhancement implies $C(p(y_{1:k}|x_{1:k})) \subseteq C(p(z_{1:k}|x_{1:k}))$

Genie-Aided Proofs: Concept & Applications

Genie design principles

- *Usefulness:* $C(p(z_{1:k}|x_{1:k}))$ is known, or admits an outer bound.
- *Smartness:* the genie does not inflate the relevant boundary (often by becoming redundant under optimal signaling in a regime):

$$C(p(y_{1:k}|x_{1:k})) = C(p(z_{1:k}|x_{1:k})) \qquad \text{or} \qquad C(p(y_{1:k}|x_{1:k})) \approx C(p(z_{1:k}|x_{1:k}))$$

Historical Developments (Interference Channel)

- **Deterministic interference (1982):** El Gamal and Costa used side information to determine capacity for a class of deterministic interference channels.
- **Gaussian outer bounds (2004):** Kramer derived tighter outer bounds via genie-aided arguments.
- **Capacity within one bit (2008):** Etkin–Tse–Wang approximated Gaussian IC capacity within one bit.
- **TIN optimality (weak interference):** Shang–Kramer–Chen (2009); Motahari–Khandani (2009); Annapureddy–Veeravalli (2009).

Other Notable Genie-Aided / Enhanced Receivers

Selected applications

- **MIMO broadcast channels:** Marton's inner bound is tight for private messages only (Weingarten, Steinberg, and Shamai 2006).
- **MAC & two-way channels:** *dependence balance bounds*; augment Y to (Y, Z) to create parallel channels (Hekstra and Willems, 1989).

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- Joint work with Chandra Nair (2022): new outer bounds for **relay**, **broadcast**, and **interference** channels.
  - Improved relay bounds (with Abbas El Gamal and Chandra Nair).
  - Improved interference bounds (with Jinpei Zhao and Chandra Nair).
  - Improved broadcast bounds (with Yi Liu and Chandra Nair).
  - Follow-up by Yanlin Geng on the BSSC channel.
- New bound for **distributed hypothesis testing** (with Zhenduo Wen).
- **Dependence balance bounds for arbitrary networks** (with Gerhard Kramer).

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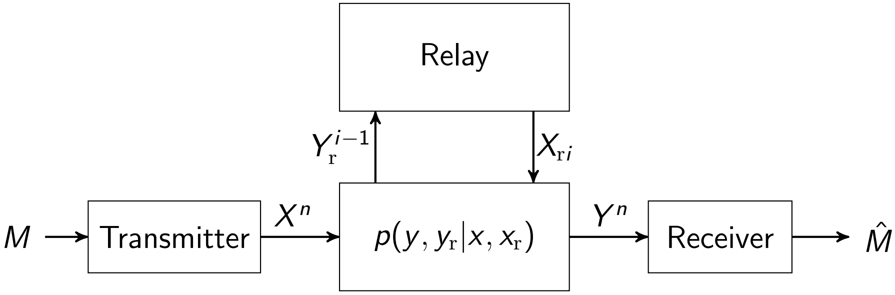

X: Transmitter's input

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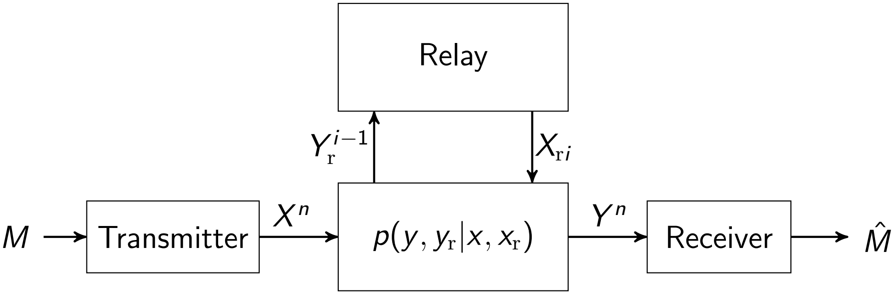

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- Goal: Reliably communicate message $M \in [1, 2^{nR}]$ from X to Y with the help of relay node (X_r, Y_r)

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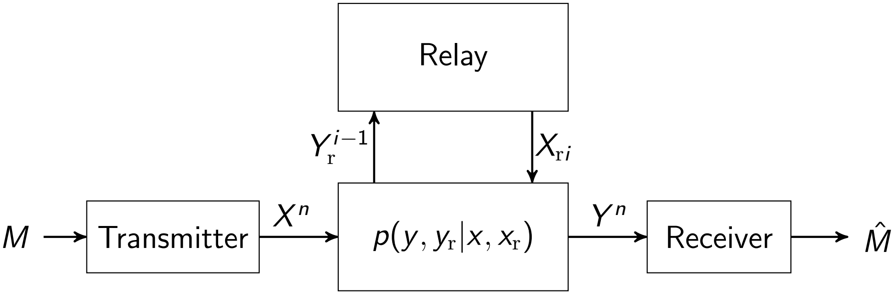

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- Goal: Reliably communicate message $M \in [1, 2^{nR}]$ from X to Y with the help of relay node $(X_{\mathrm{r}}, Y_{\mathrm{r}})$
- Relay transmission: $X_{\mathrm{r}i} = f_i(Y_{\mathrm{r}1}, Y_{\mathrm{r}2}, \cdots, Y_{\mathrm{r}(i-1)})$

Cutset upper bound

- Upper bound on the capacity of relay channel (Cover and El Gamal 1979):

  $$\max_{p(x,x_{\mathrm{r}})} \min \left\{ I(X, X_{\mathrm{r}}; Y),\ I(X; Y, Y_{\mathrm{r}} | X_{\mathrm{r}}) \right\}$$

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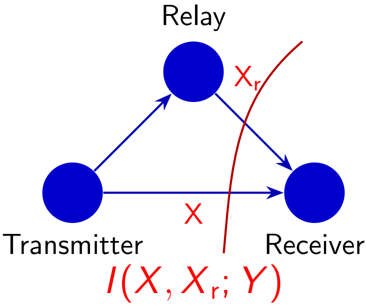
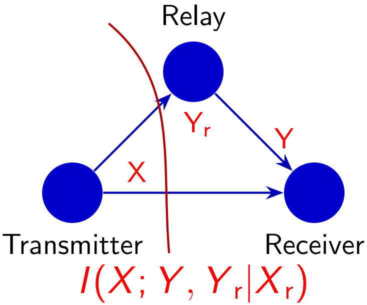
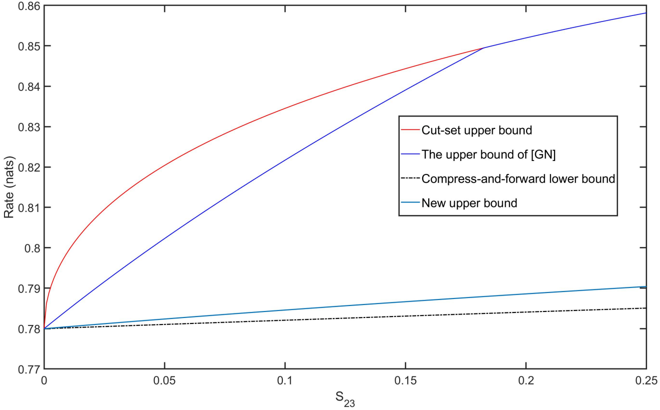
- Max-flow Min-cut interpretation:





Relay Channel: Improved Outer Bounds

- Joint work with C. Nair (2022) established strict improvement over the cut-set bound for the scalar Gaussian case. Better bounds in follow-up paper with A. El Gamal and C. Nair.



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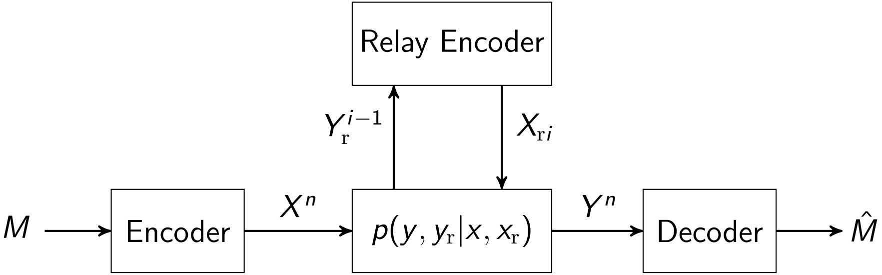
- A dependence-balance outer bound for the relay channel (with Gerhard Kramer)



Theorem (Dependence-balance outer bound)

For any arbitrary $P_{Z|X,X_r,Y,Y_r}$, we have

$$R \leq \min\left[I(X;Y,Y_r|X_r), I(X,X_r;Y)\right]$$
$$R \leq \min\left[I(X;Y,Y_r,Z|X_r,T), I(X,X_r;Y,Z|T)\right]$$

for some $p_{X_1,X_r,T}$ satisfying $I(X;X_r|T) \leq I(X;X_r,Y_r|T,Z)$.

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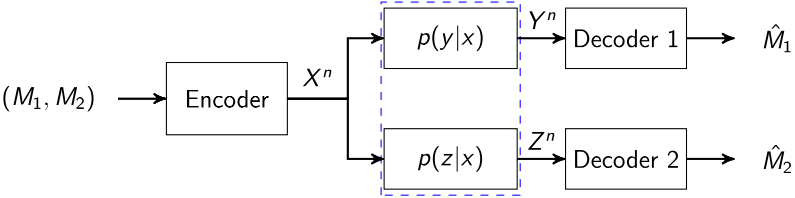
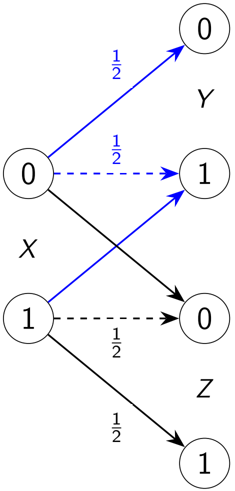


- The UV outer bound (Nair–El Gamal, 2006) can be improved using auxiliary receivers.
- Generalizes an earlier outer bound for product broadcast channels.
- Yields tight bounds for sum of broadcast channels with known capacity (e.g., semi-deterministic, more capable).

Binary Skew-Symmetric Broadcast Channel (BSSC)



- UV outer bound implies $R_1 + R_2 \leq 0.37255$ (Nair–El Gamal '06).
- Marton's inner bound achieves $R_1 + R_2 \approx 0.3616$.
- Yanlin Geng: auxiliary choice yields $R_1 + R_2 \leq 0.3693$.
- With two auxiliary receivers: improve to 0.3692 (joint work with Liu and Nair).

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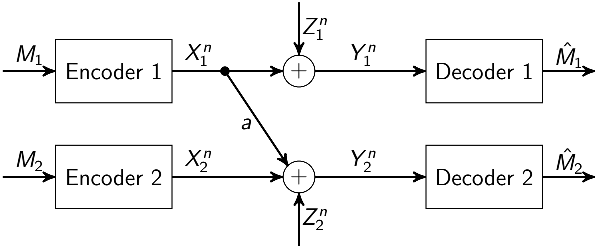
Gaussian Z-Interference Channel



$$\begin{aligned} Y_1 &= X_1 + Z_1 \\ Y_2 &= aX_1 + X_2 + Z_2, \end{aligned}$$

where $a \in (0,1)$, $Z_i \sim \mathcal{N}(0,1)$. In addition, a power constraint is imposed on the n-letter codes:

$$E\|X_i^n\|^2 \leq nP_i, \quad i = 1, 2.$$

Costa's corner point: max R_1 such that (R_1, C_2) is achievable.

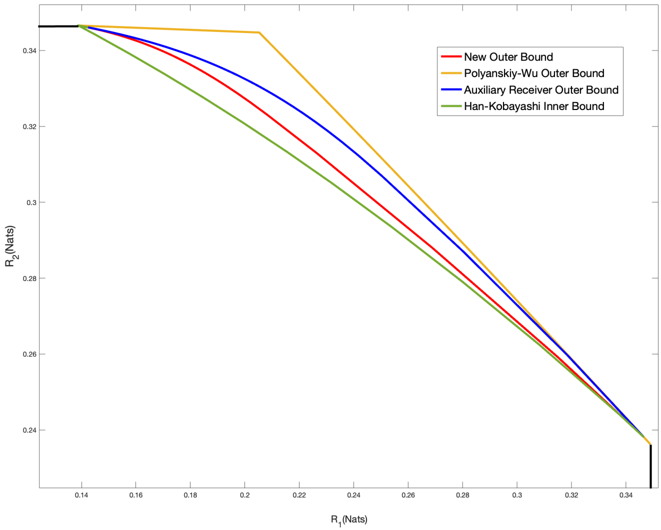
Costa's Corner Point Problem

- Costa (1985): Initial proof of corner point
- Sason (2004): Identified flaw in proof
- Polyanskiy-Wu (2015): Fixed using Talagrand's inequality to get a continuity of entropy argument
- Open question: Is corner point exposed or extreme?

Auxiliary Receiver Solution

- Recovers Costa's corner point: $R_1^* = \frac{1}{2} \log \left(1 + \frac{a^2 P_1}{1+P_2}\right)$
- Shows corner point is **exposed** (kink in capacity region)
- Strict improvement over existing bounds

Here $a = 0.8, P_1 = P_2 = 1$.



General Gaussian Interference Channel

$$
\begin{aligned}
Y_1 &= X_1 + bX_2 + Z_1 \\
Y_2 &= X_2 + aX_1 + Z_2
\end{aligned}
$$

- Power constraints: $\mathbb{E}[X_1^2] \le P_1$, $\mathbb{E}[X_2^2] \le P_2$.
- Sum-capacity known when $a(1 + b^2 P_2) + b(1 + a^2 P_1) < 1$ (very weak interference) Shang–Kramer–Chen (2009); Motahari–Khandani (2009); Annapureddy–Veeravalli (2009).

With Chandra Nair and Jinpei Zhao, we develop an outer bound that applies to general discrete memoryless channels, passes through the Costa corner points, and is tight in the very weak interference regime.

Outline

1. Applications of the Auxiliary Receivers
2. Why Do Auxiliary Receivers Help?
   - Relay Channel
   - Broadcast Channel
   - Distributed Hypothesis Testing
3. Conclusion

The Capacity Region Surface

- Consider communication over a general communication network $p(y_{1:k}|x_{1:k})$ with k nodes.
- Capacity region $\mathsf{C}\big(p(y_{1:k}|x_{1:k})\big)$ as a function of $p(y_{1:k}|x_{1:k})$.

Space of all Channels $p(\cdot|\cdot)$

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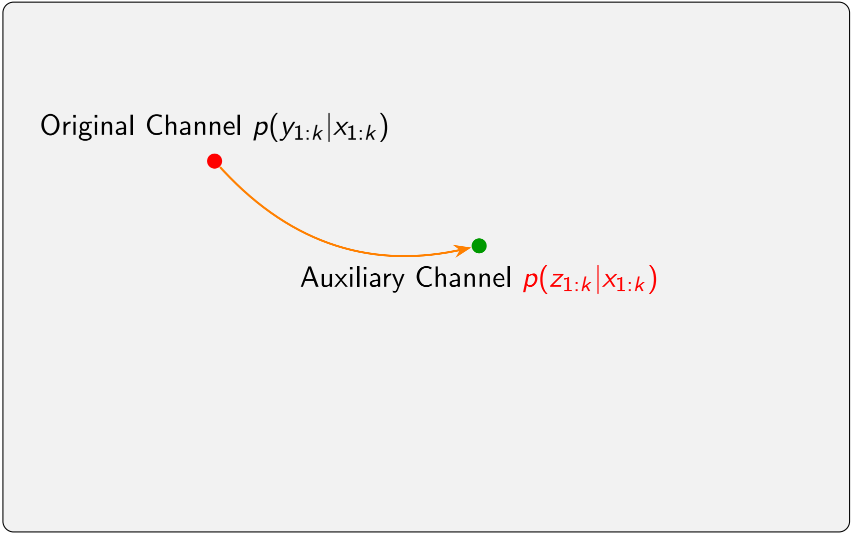
Original Channel $p(y_{1:k}|x_{1:k})$

Space of all Channels $p(\cdot|\cdot)$



Original Channel $p(y_{1:k}|x_{1:k})$

Auxiliary Channel $p(z_{1:k}|x_{1:k})$

Deriving Outer Bounds via Auxiliary Channels

- In general, the channel $p(z_{1:k}|x_{1:k})$ need not be an enhancement of $p(y_{1:k}|x_{1:k})$.
- Auxiliary receivers can be used to study the **surface** and **curvature** of this region with respect to variations in the channel $p(y_{1:k}|x_{1:k})$.

An outer bound on $\mathsf{C}\big(p(y_{1:k}|x_{1:k})\big)$ can be derived if two conditions are met:

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Deriving Outer Bounds via Auxiliary Channels

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2. A suitable outer bound on the auxiliary region $\mathsf{C}\big(p(z_{1:k}|x_{1:k})\big)$ is available.

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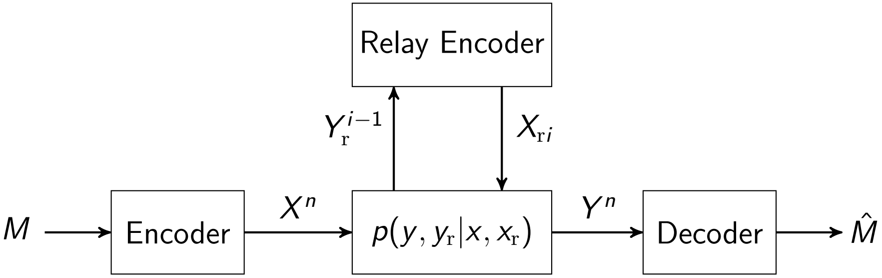
Illustration in the relay channel problem



Main channel

$$p(y_{\mathrm{r}}|x, x_{\mathrm{r}})p(y|y_{\mathrm{r}}, x, x_{\mathrm{r}})$$

Auxiliary channel

$$p(y_{\mathrm{r}}|x, x_{\mathrm{r}})p(z|x)$$

$$\begin{aligned} C(P_{Y,Y_{\mathrm{r}}|X,X_{\mathrm{r}}}) &= C(P_{Z,Y_{\mathrm{r}}|X,X_{\mathrm{r}}}) + C(P_{Y,Y_{\mathrm{r}}|X,X_{\mathrm{r}}}) - C(P_{Z,Y_{\mathrm{r}}|X,X_{\mathrm{r}}}) \\ &\leq I(X; Z, Y_{\mathrm{r}}|X_{\mathrm{r}}) + C(P_{Y,Y_{\mathrm{r}}|X,X_{\mathrm{r}}}) - C(P_{Z,Y_{\mathrm{r}}|X,X_{\mathrm{r}}}) \end{aligned}$$

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Recap:

$$C(P_{Y,Y_{\mathrm{r}}|X,X_{\mathrm{r}}}) = \sup_{n, P_{M,X^n}, P_{X_{\mathrm{r}i}|Y_{\mathrm{r}}^{i-1}}} \frac{1}{n} I(M; Y^n)$$

$$\begin{aligned} C(P_{Y,Y_{\mathrm{r}}|X,X_{\mathrm{r}}}) &= C(P_{Z,Y_{\mathrm{r}}|X,X_{\mathrm{r}}}) + C(P_{Y,Y_{\mathrm{r}}|X,X_{\mathrm{r}}}) - C(P_{Z,Y_{\mathrm{r}}|X,X_{\mathrm{r}}}) \\ &\le I(X;Z,Y_{\mathrm{r}}|X_{\mathrm{r}}) + C(P_{Y,Y_{\mathrm{r}}|X,X_{\mathrm{r}}}) - C(P_{Z,Y_{\mathrm{r}}|X,X_{\mathrm{r}}}) \end{aligned}$$

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- Using the multi-letter characterizations

$$\begin{aligned} & C(P_{Y,Y_{\mathrm{r}}|X,X_{\mathrm{r}}}) - C(P_{Z,Y_{\mathrm{r}}|X,X_{\mathrm{r}}}) \\ &= \sup_{\text{codes}} \frac{1}{n} I(M;Y^n) - \sup_{\text{codes}} \frac{1}{n} I(M;Z^n) \\ &\le \sup_{\text{codes}} \frac{1}{n} \left(I(M;Y^n) - I(M;Z^n) \right) \end{aligned}$$

Single-letterization of the Difference Term

$$\begin{aligned}\frac{1}{n}\Big(I(M;Y^n)-I(M;Z^n)\Big) &= \frac{1}{n}\sum_{i=1}^{n}\Big(I(M;Y^iZ_{i+1}^n)-I(M;Y^{i-1}Z_i^n)\Big)\\ &= \frac{1}{n}\sum_{i=1}^{n}\Big(I(M;Y_i|Y^{i-1}Z_{i+1}^n)-I(M;Z_i|Y^{i-1}Z_{i+1}^n)\Big)\\ &\leq \frac{1}{n}\sum_{i=1}^{n}\Big(I(X_i,X_{\mathrm{r}i};Y_i|U_i)-I(X_i,X_{\mathrm{r}i};Z_i|U_i)\Big),\end{aligned}$$

where $U_i=(Y^{i-1},Z_{i+1}^n)$ satisfies

$$U_i\to(X_{\mathrm{r}i},X_i)\to(Y_i,Z_i,Y_{\mathrm{r}i}).$$

The last step uses the auxiliary channel structure

$$p(y_\mathrm{r}|x,x_\mathrm{r})\,p(z|x).$$

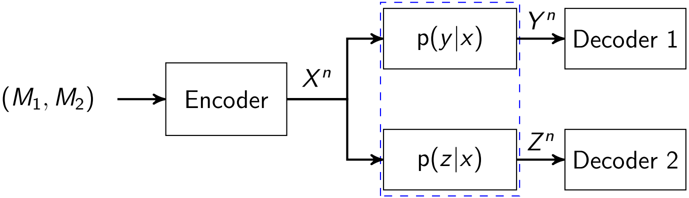
$$\begin{aligned} C(P_{Y,Y_{\mathrm{r}}|X,X_{\mathrm{r}}}) &= C(P_{Z,Y_{\mathrm{r}}|X,X_{\mathrm{r}}}) + C(P_{Y,Y_{\mathrm{r}}|X,X_{\mathrm{r}}}) - C(P_{Z,Y_{\mathrm{r}}|X,X_{\mathrm{r}}}) \\ &\leq I(X;Z,Y_{\mathrm{r}}|X_{\mathrm{r}}) + I(X,X_{\mathrm{r}};Y|U) - I(X,X_{\mathrm{r}};Z|U) \end{aligned}$$

for some

$$U \to (X_{\mathrm{r}}, X) \to (Y, Z, Y_{\mathrm{r}}).$$

Outline

Goal & Setting



Let $p_{Y,Z|X}$ be a memoryless broadcast channel and fix an input distribution p_X.
For illustration, we only look at the sum-rate. Define $\mathrm{SR}(p_{Y,Z|X})$ as the maximum achievable sum-rate $R_1 + R_2$ subject to the empirical input distribution converging to p_X.

Multi-letter Characterization

$$\mathrm{SR}(p_{Y,Z|X}) = \sup_{\mathcal{Q}} \frac{1}{n}\Big(I(M_1; Y^n) + I(M_2; Z^n)\Big).$$

Define $\mathcal{Q}_n$ as all distributions

$$p_{M_1,M_2,X^n} = p_{M_1}\, p_{M_2}\, p_{X^n|M_1,M_2},$$

with arbitrary message alphabets, such that the induced p_{X^n} satisfies

$$\frac{1}{n}\sum_{i=1}^{n} p_{X_i}(x) = p_X(x).$$

Let $\mathcal{Q} \triangleq \bigcup_{n\geq 1} \mathcal{Q}_n$.

Special Case: $Y = Z = J$

If $Y = Z = J$, receiver J can decode both messages, hence

$$\mathrm{SR}(p_{J,J|X}) = I(X; J).$$

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An “apparently better” multi-letter identity:

$$\mathrm{SR}(p_{J,J|X}) = \sup_{\mathcal{Q}} \frac{1}{n} \left(I(M_1; J^n) + I(M_2; J^n) + \sum_{i=1}^{n} I(J_i; J^{i-1}) \right).$$

Proof Sketch

For every $P_{M_1,M_2,X^n} \in \mathcal{Q}_n$,

$$\begin{aligned}
& I(M_1; J^n) + I(M_2; J^n) + \sum_{i=1}^{n} I(J_i; J^{i-1}) \\
& \leq I(M_1, M_2; J^n) + \sum_{i=1}^{n} I(J_i; J^{i-1}) \\
& \leq I(X^n; J^n) + \sum_{i=1}^{n} I(J_i; J^{i-1}) \\
& = \sum_{i=1}^{n} I(X_i; J_i) \leq n\, I(X; J),
\end{aligned}$$

using memorylessness, the fixed-input constraint, and concavity of mutual information in p_X.

Add-and-Subtract (Auxiliary Receiver Approach)

$$\begin{aligned}\mathrm{SR}(p_{Y,Z|X}) &= \mathrm{SR}(p_{J,J|X}) + \big(\mathrm{SR}(p_{Y,Z|X}) - \mathrm{SR}(p_{J,J|X})\big)\\ &= I(X;J) + \mathrm{SR}(p_{Y,Z|X}) - \mathrm{SR}(p_{J,J|X}).\end{aligned}$$

Bounding the Difference of Suprema

$$
\begin{aligned}
&\mathrm{SR}\big(p_{Y,Z|X}\big) - \mathrm{SR}\big(p_{J,J|X}\big) \\
&= \sup_{\text{codes}} \frac{1}{n}\big(I(M_1; Y^n) + I(M_2; Z^n)\big) \\
&\qquad - \sup_{\text{codes}} \frac{1}{n}\left(I(M_1; J^n) + I(M_2; J^n) + \sum_{i=1}^{n} I(J_i; J^{i-1})\right) \\
&\le \sup_{\text{codes}} \frac{1}{n}\Big(I(M_1; Y^n) + I(M_2; Z^n) - I(M_1; J^n) - I(M_2; J^n) \\
&\qquad\qquad - \sum_{i=1}^{n} I(J_i; J^{i-1})\Big).
\end{aligned}
$$

Telescoping Expansions

Observe that

$$I(M_1; Y^n) - I(M_1; J^n) = \sum_{i=1}^{n} \Big(I(\hat{U}_i; Y_i|\hat{W}_i) - I(\hat{U}_i; J_i|\hat{W}_i) \Big),$$

$$I(M_2; Z^n) - I(M_2; J^n) = \sum_{i=1}^{n} \Big(I(\tilde{V}_i; Z_i|\tilde{W}_i) - I(\tilde{V}_i; J_i|\tilde{W}_i) \Big),$$

where

$$\hat{U}_i = \tilde{U}_i = M_1, \quad \hat{V}_i = \tilde{V}_i = M_2,$$

and

$$\hat{W}_i = (Y^{i-1}, J_{i+1}^n), \qquad \tilde{W}_i = (Z^{i-1}, J_{i+1}^n).$$

Manipulating the Memory Term

$$\hat{W}_i = (Y^{i-1}, J_{i+1}^n), \qquad \tilde{W}_i = (Z^{i-1}, J_{i+1}^n).$$

The term $-\sum_{i=1}^n I(J_i; J^{i-1})$ can be bounded as

$$\begin{aligned} -\sum_{i=1}^n I(J_i; J^{i-1}) &\leq -\sum_{i=1}^n I(J_i; J^{i-1}) + \sum_{i=1}^n I(Y_i; Y^{i-1}) \\ &= \sum_{i=1}^n \Big(I(\hat{W}_i; Y_i) - I(\hat{W}_i; J_i) \Big). \end{aligned}$$

Similarly,

$$-\sum_{i=1}^n I(J_i; J^{i-1}) \leq \sum_{i=1}^n \Big(I(\tilde{W}_i; Z_i) - I(\tilde{W}_i; J_i) \Big).$$

Time-sharing and the Final J-Bound Upper Bound

Introduce a time-sharing random variable Q and define

$$\tilde{W} = (\tilde{W}_Q, Q), \qquad \hat{W} = (\hat{W}_Q, Q),$$

(and similarly for $\hat{U}$, $\tilde{V}$, etc.). Then we obtain the upper bound:

$$\begin{aligned}
\mathrm{SR}\big(p_{Y,Z|X}\big) \leq I(X;J) &+ I(\hat{U};Y|\hat{W}) - I(\hat{U};J|\hat{W}) \\
&+ I(\tilde{V};Z|\tilde{W}) - I(\tilde{V};J|\tilde{W}) \\
&+ \min\Big(I(\hat{W};Y) - I(\hat{W};J),\ I(\tilde{W};Z) - I(\tilde{W};J)\Big),
\end{aligned}$$

for some $p_{\hat{U},\hat{W},\tilde{U},\tilde{W},X}\, p_{J,Y,Z|X}$.

Relation to UVW Outer Bounds

$$\begin{aligned}\mathrm{SR}\big(p_{Y,Z|X}\big) \leq\ & I(X;J) + I(\hat{U};Y|\hat{W}) - I(\hat{U};J|\hat{W}) \\ & + I(\tilde{V};Z|\tilde{W}) - I(\tilde{V};J|\tilde{W}) \\ & + \min\Big(I(\hat{W};Y) - I(\hat{W};J),\ I(\tilde{W};Z) - I(\tilde{W};J)\Big),\end{aligned}$$

for some $p_{\hat{U},\hat{W},\tilde{U},\tilde{W},X}\, p_{J,Y,Z|X}$.

This upper bound is valid for any choice of $p_{J|X,Y,Z}$ (i.e., any auxiliary J).

- Setting $J = Y$ or $J = Z$ yields the two UVW outer bound sum-rate constraints.
- Choosing $J \notin \{Y, Z\}$ can yield strict improvements over the UVW outer bound.

Further improvement

Move from

$$\mathrm{SR}\big(p_{Y,Z|X}\big) \text{ to } \mathrm{SR}\big(p_{G,K|X}\big)\,,$$

allowing for even better bounds than moving to $\mathrm{SR}\big(p_{J,J|X}\big)$.

$$\begin{tikzcd}
 & & p_{G,G|X} \\
p_{Y,Z|X} \arrow[r] & p_{G,K|X} \arrow[ur] \arrow[dr] & \\
 & & p_{K,K|X}
\end{tikzcd}$$

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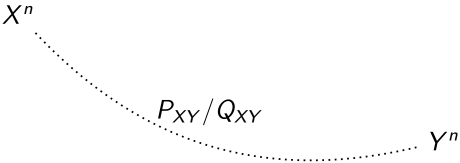
Problem Setup

Two terminals observe n i.i.d. repetitions of X and Y:

$$(X, Y) \sim \begin{cases} P_{XY}, & \text{under } \mathcal{H}_0, \\ Q_{XY}, & \text{under } \mathcal{H}_1. \end{cases}$$



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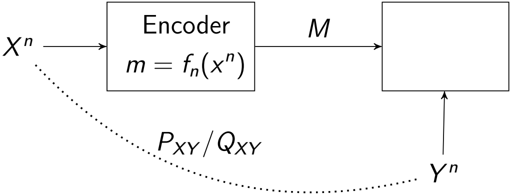

- $f_n : \mathcal{X}^n \to \mathcal{M} = \{1, 2, \dots, 2^{\lfloor nR \rfloor}\}$ is sent to the second terminal.

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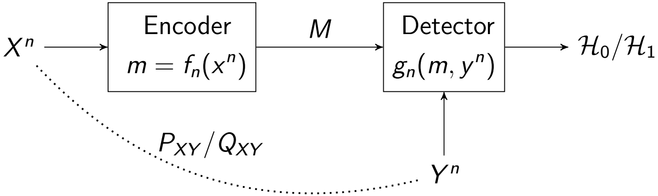

- $f_n : \mathcal{X}^n \to \mathcal{M} = \{1, 2, \ldots, 2^{\lfloor nR \rfloor}\}$ is sent to the second terminal.
- The detector decides between the hypotheses $\mathcal{H}_0$ and $\mathcal{H}_1$.

Error Exponent

Definition

Given blocklength n and rate R,

$$E_{P_{XY},Q_{XY}}(R) \triangleq \lim_{\epsilon\to 0} \lim_{n\to\infty} -\frac{1}{n}\log p(\epsilon, n, R),$$

where $p(\epsilon, n, R)$ is the smallest achievable type-II error probability under $\mathcal{H}_1$ subject to type-I error $\leq \epsilon$ under $\mathcal{H}_0$.

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In general, $E_{P_{XY},Q_{XY}}(R)$ is unknown.

Known Results

- Upper bounds: Ahlswede–Csiszár (1986), Shalaby–Papamarcou (1992), Rahman–Wagner (2012), Hadar–Liu–Polyanskiy–Shayevitz (2019), Kochman (2024).
- Lower bounds: Ahlswede–Csiszár (1986), Han (1987), Shimokawa–Han–Amari (1994).

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$$E_{P_{XY},Q_{XY}}(R) \leq E_{P_{XY},Q_{XY}}(\infty) = D(P_{XY}\|Q_{XY}).$$

- Multi-letter characterization (Ahlswede–Csiszár, 1986):

$$E_{P_{XY},Q_{XY}}(R) = \sup_{n,\, P_{M|X^n}} \frac{1}{n} D(P_{MY^n}\|Q_{MY^n}),$$

where $M \in \{1, \dots, 2^{\lfloor nR \rfloor}\}$.

Add-and-subtract manipulation

For any $Q_{Z|X}, P_{Z|X}$:

$$\begin{aligned}
&E(P_{XY}, Q_{XY}, R)\\
&= E(P_{XY}, Q_{XY}, R) - E(P_{XZ}, Q_{XZ}, R) + E(P_{XZ}, Q_{XZ}, R)\\
&\leq E(P_{XY}, Q_{XY}, R) - E(P_{XZ}, Q_{XZ}, R) + D(P_{XZ} \| Q_{XZ})
\end{aligned}$$

Using multi-letter forms:

$$\begin{aligned}
&E_{P_{XY},Q_{XY}}(R) - E_{P_{XZ},Q_{XZ}}(R)\\
&= \sup_{n, P_{M|X^n}} \frac{1}{n} D(P_{MY^n} \| Q_{MY^n}) - \sup_{n, P_{M|X^n}} \frac{1}{n} D(P_{MZ^n} \| Q_{MZ^n})\\
&\leq \sup_{n, P_{M|X^n}} \left\{ \frac{1}{n} D(P_{MY^n} \| Q_{MY^n}) - D(P_{MZ^n} \| Q_{MZ^n}) \right\}
\end{aligned}$$

Telescopic Single-letterization

Expand the difference term via a telescoping sum:

$$\begin{aligned}
&\frac{1}{n}\Big(D(P_{MY^n}\|Q_{MY^n}) - D(P_{MZ^n}\|Q_{MZ^n})\Big)\\
&= \frac{1}{n}\sum_{i=1}^{n}\left\{D(P_{MY_{1:i-1}Y_iZ_{i+1:n}}\|Q_{MY_{1:i-1}Y_iZ_{i+1:n}}) - D(P_{MY_{1:i-1}Z_iZ_{i+1:n}}\|Q_{MY_{1:i-1}Z_iZ_{i+1:n}})\right\}\\
&= \frac{1}{n}\sum_{i=1}^{n}\Big(D(P_{U_iY_i}\|Q_{U_iY_i}) - D(P_{U_iZ_i}\|Q_{U_iZ_i})\Big),
\end{aligned}$$

where $U_i = (M, Y_{1:i-1}, Z_{i+1:n})$.

Adding Markov constraints

The auxiliary

$$U_i = (M, Y_{1:i-1}, Z_{i+1:n}) \quad \text{satisfies} \quad U_i \to X_i \to (Y_i, Z_i).$$

Rate constraints imply (schematically):

$$\sum_i I_P(U_i; X_i | Y_i) \le R, \quad \sum_i I_P(U_i; X_i | Z_i) \le R,$$
$$\sum_i I_Q(U_i; X_i | Y_i) \le R, \quad \sum_i I_Q(U_i; X_i | Z_i) \le R.$$

Thus,

$$E(P_{XY}, Q_{XY}, R) - E(P_{XZ}, Q_{XZ}, R) \le \sup \Big(D(P_{UY} \| Q_{UY}) - D(P_{UZ} \| Q_{UZ}) \Big),$$

over $P_{U|X}, Q_{U|X}$ satisfying the corresponding mutual-information constraints.

Gaussian Distributed Hypothesis Testing

Under H_0:

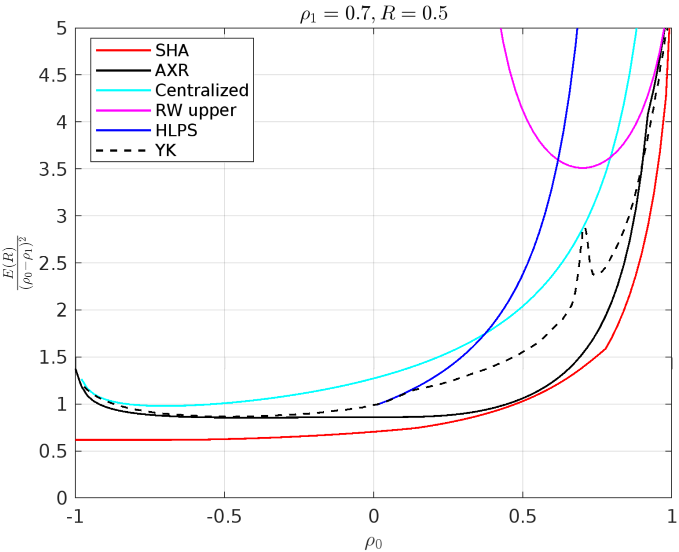
$$(X, Y) \sim \mathcal{N}\left(0, \begin{bmatrix} 1 & \rho_0 \\ \rho_0 & 1 \end{bmatrix}\right)$$

Under H_1:

$$(X, Y) \sim \mathcal{N}\left(0, \begin{bmatrix} 1 & \rho_1 \\ \rho_1 & 1 \end{bmatrix}\right)$$

In the following plot, denote our auxiliary receiver bound as AXR, quantization-and-binning lower bound by Shimokawa, Han and Amari as SHA, Hadar et al. as HLPS, Rahman and Wagner's upper bound by RW upper, and Kochman's upper bound by YK.

Comparison of bounds



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Summary

- **Versatility:** Auxiliary variables (receivers/remote sources) are powerful tools for deriving converse bounds.
- **Unification:** The framework unifies genie-aided bounds, remote sources, and channel enhancement. It uses systematic manipulations.
- **Impact:** Leads to improved bounds in:
  - Relay, Broadcast and Interference Channels
  - Dependence-balance bounds for general networks
  - Distributed Hypothesis Testing
- **Future Directions:**
  - Sequences of auxiliary receivers (Kochman 2024).
  - Applications in Quantum Information Theory?

Thank You