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Random purification channel made simple

Filippo Girardi
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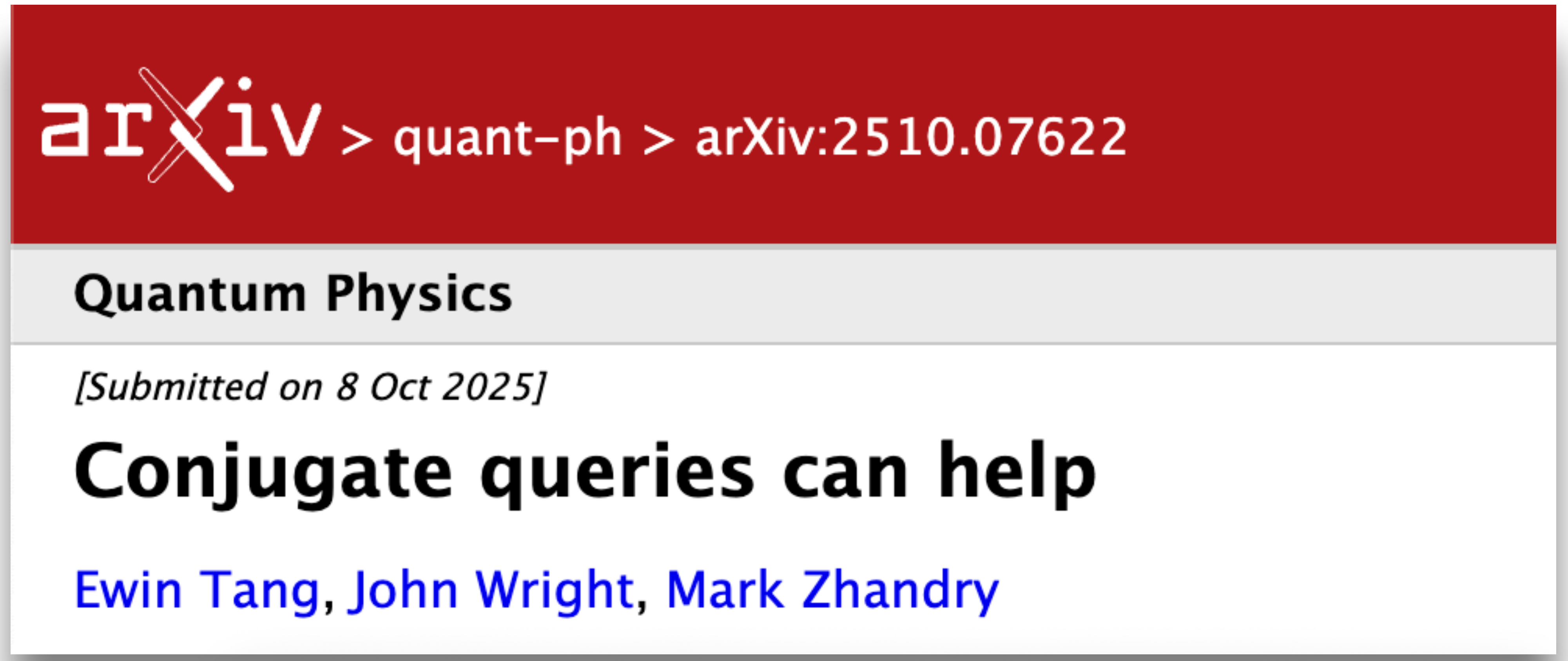


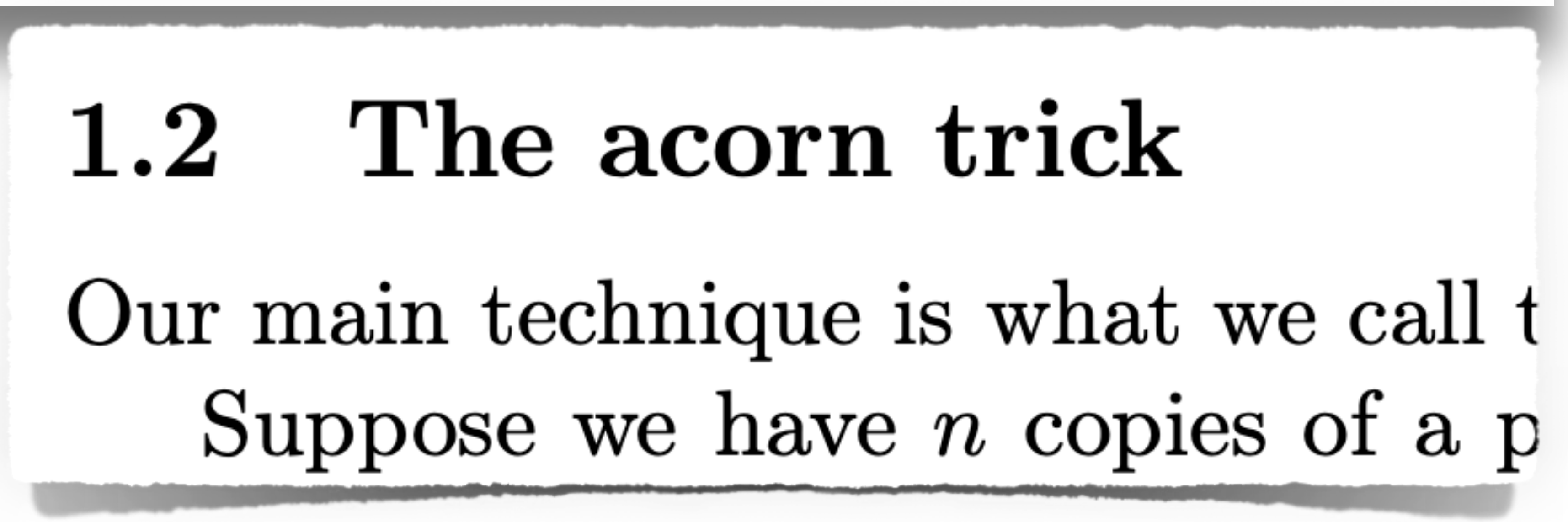
arXiv > quant-ph > arXiv:2510.07622

Quantum Physics

[Submitted on 8 Oct 2025]

Conjugate queries can help

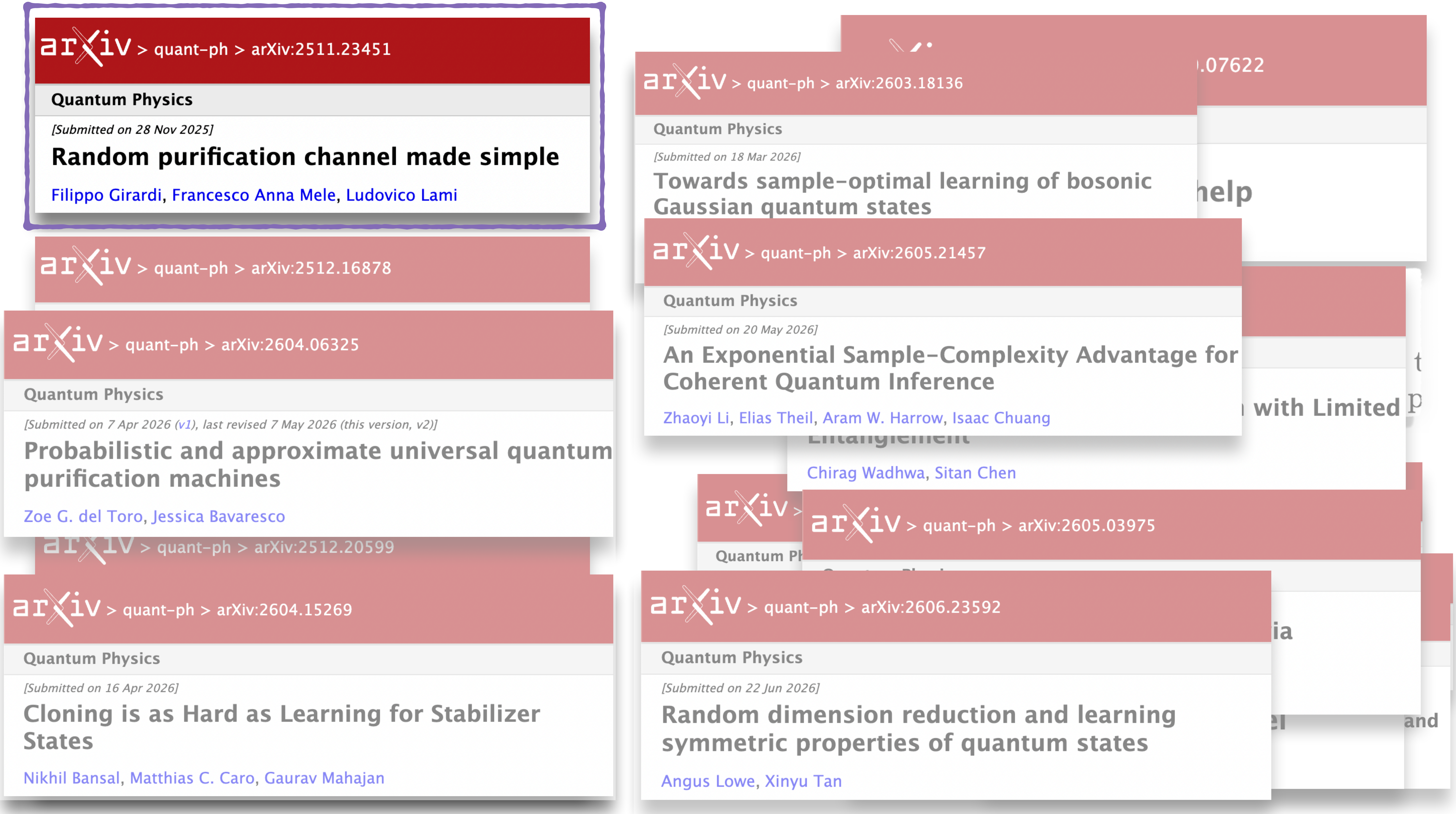
Ewin Tang, John Wright, Mark Zhandry



1.2 The acorn trick

Our main technique is what we call t

Suppose we have n copies of a p



arXiv > quant-ph > arXiv:2511.23451

Quantum Physics

[Submitted on 28 Nov 2025]

Random purification channel made simple

Filippo Girardi, Francesco Anna Mele, Ludovico Lami

arXiv > quant-ph > arXiv:2512.16878

arXiv > quant-ph > arXiv:2604.06325

Quantum Physics

[Submitted on 7 Apr 2026 (v1), last revised 7 May 2026 (this version, v2)]

Probabilistic and approximate universal quantum purification machines

Zoe G. del Toro, Jessica Bavaresco

arXiv > quant-ph > arXiv:2512.20599

arXiv > quant-ph > arXiv:2604.15269

Quantum Physics

[Submitted on 16 Apr 2026]

Cloning is as Hard as Learning for Stabilizer States

Nikhil Bansal, Matthias C. Caro, Gaurav Mahajan

.07622

help

arXiv > quant-ph > arXiv:2603.18136

Quantum Physics

[Submitted on 18 Mar 2026]

Towards sample-optimal learning of bosonic Gaussian quantum states

arXiv > quant-ph > arXiv:2605.21457

Quantum Physics

[Submitted on 20 May 2026]

An Exponential Sample-Complexity Advantage for Coherent Quantum Inference

Zhaoyi Li, Elias Theil, Aram W. Harrow, Isaac Chuang

with Limited

Entanglement

Chirag Wadhwa, Sitan Chen

arXiv >

Quantum Ph

arXiv > quant-ph > arXiv:2605.03975

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and

arXiv > quant-ph > arXiv:2606.23592

Quantum Physics

[Submitted on 22 Jun 2026]

Random dimension reduction and learning symmetric properties of quantum states

Angus Lowe, Xinyu Tan

Overview

$$\rho_A^{\otimes N} \xrightarrow{\quad} |\psi_\rho\rangle_{AE}^{\otimes N}$$

Why do we want to purify states?

(and why this is physically impossible)

How can we make this possible?

(random purification channel)

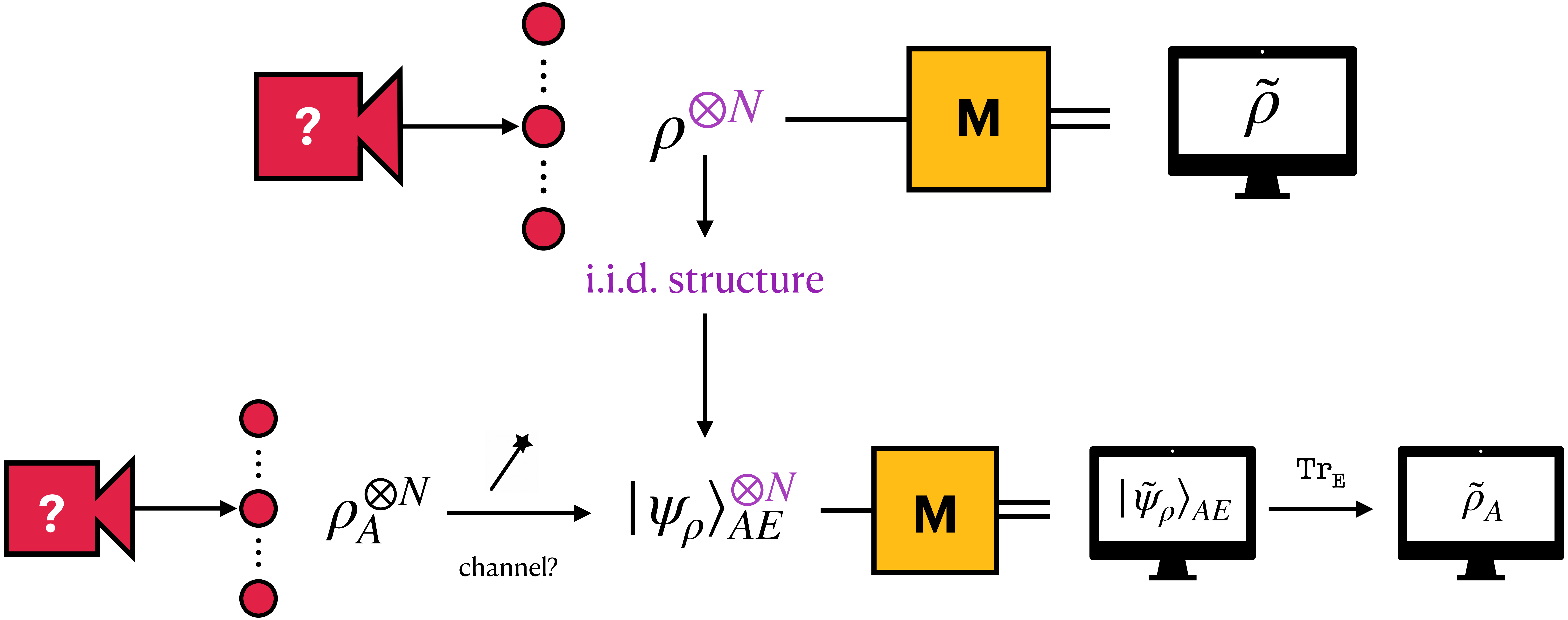
→ the simplest construction of the random purification channel

(without representation theory)

→ applications and generalisations

(new tool in quantum learning theory, but also in quantum Shannon theory)

Learning theory



?

$\rho^{\otimes N}$

M

$\tilde{\rho}$

i.i.d. structure

?

$\rho_A^{\otimes N}$

channel?

$|\psi_\rho\rangle_{AE}^{\otimes N}$

M

$|\tilde{\psi}_\rho\rangle_{AE}$

Tr_E

$\tilde{\rho}_A$

No purifying channel

Physical implementation of a **universal purification channel** $\tilde{\Lambda}$?

$$\rho_A \xmapsto{\tilde{\Lambda}} |\rho\rangle_{AB} \qquad\qquad \sigma_A \xmapsto{\tilde{\Lambda}} |\sigma\rangle_{AB}$$

$$\tilde{\Lambda}(\rho_A) = |\rho\rangle\langle\rho|_{AB} \qquad\qquad \tilde{\Lambda}(\sigma_A) = |\sigma\rangle\langle\sigma|_{AB}$$

linearity

$$\tilde{\Lambda}\left(\frac{1}{2}\rho_A + \frac{1}{2}\sigma_A\right) = \frac{1}{2}|\rho\rangle\langle\rho|_{AB} + \frac{1}{2}|\sigma\rangle\langle\sigma|_{AB}$$

mixed (unless $|\rho\rangle = |\sigma\rangle$)

Z. Liu, Z. Du, Z. Cai, and Z.-W. Liu. *No universal purification in quantum mechanics*. Preprint arXiv:2509.21111 (2025)
K. Fang and Z.-W. Liu. *No-go theorems for quantum resource purification*. Phys. Rev. Lett., 125(6):060405, (2020)
K. Fang and Z.-W. Liu. *No-go theorems for quantum resource purification: New approach and channel theory*. PRX Quantum, 3(1):010337 (2022)

Haar measure

We denote by

$$\mathbb{E}_U\big[f(U)\big] := \int_{\mathrm{U}(d)} f(U)\, d\mu_H(U)$$

the expectation value of f over the Haar measure on $\mathrm{U}(d) \ni U$,

where μ_H is the **Haar measure** on the unitary group $\mathrm{U}(d)$:

$$\int_{\mathrm{U}(d)} f(U)\, d\mu_H(U) = \int_{\mathrm{U}(d)} f(U{\color{blue}V})\, d\mu_H(U) = \int_{\mathrm{U}(d)} f({\color{blue}V}U)\, d\mu_H(U) \qquad {\color{blue}V} \in \mathrm{U}(d)$$

Antonio Anna Mele, *Introduction to Haar Measure Tools in Quantum Information: A Beginner's Tutorial*, Quantum 8, 1340 (2024)

Random purification channel

Let $\mathscr{H}_A \simeq \mathscr{H}_B$ be two isomorphic Hilbert spaces.
For any integer $n \geq 1$, let R_n be the operator on $\left(\mathscr{H}_A \otimes \mathscr{H}_B\right)^{\otimes n}$ defined as

$$R_n := \mathbb{E}_{U_B}\left[(\mathbb{I}_{A^n} \otimes U_B^{\otimes n})\, \Gamma_{AB}^{\otimes n}\, (\mathbb{I}_{A^n} \otimes (U_B^{\dagger})^{\otimes n})\right].$$

The channel

$$\Lambda^{(n)}(\,\cdot\,) := \sqrt{R_n}\left(\,\cdot \otimes \mathbb{I}_{B^n}\right)\sqrt{R_n}$$

acts on i.i.d. states $\rho_A^{\otimes n}$, with $\rho_A \in \mathscr{H}_A$, as

$$\boxed{\Lambda^{(n)}\left(\rho_A^{\otimes n}\right) = \mathbb{E}_{U_B}\left[(\mathbb{I}_{A^n} \otimes U_B^{\otimes n})\, |\rho\rangle\langle\rho|_{AB}^{\otimes n}\, (\mathbb{I}_{A^n} \otimes U_B^{\dagger\,\otimes n})\right],}$$

where the expectation is taken over Haar-random unitaries U_B, and $|\rho\rangle\langle\rho|$ denotes any fixed purification of ρ.

FG, F. A. Mele, L. Lami. *Random purification channel made simple*. arXiv:2511.23451 (2025)

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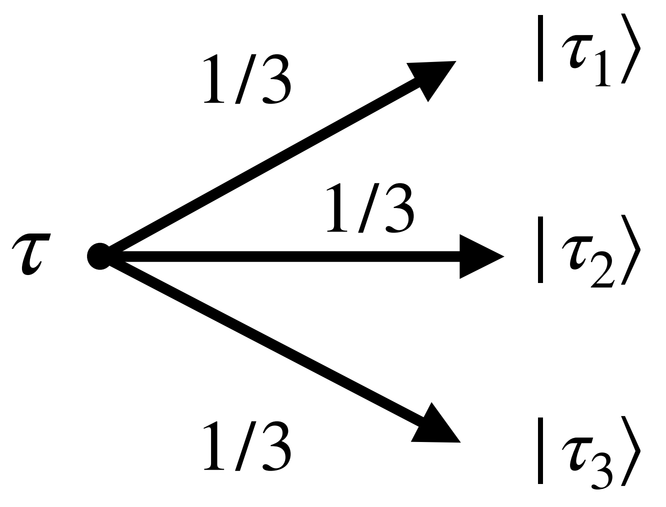
$(U_B^\dagger)^{\otimes n})\Big]$.

The

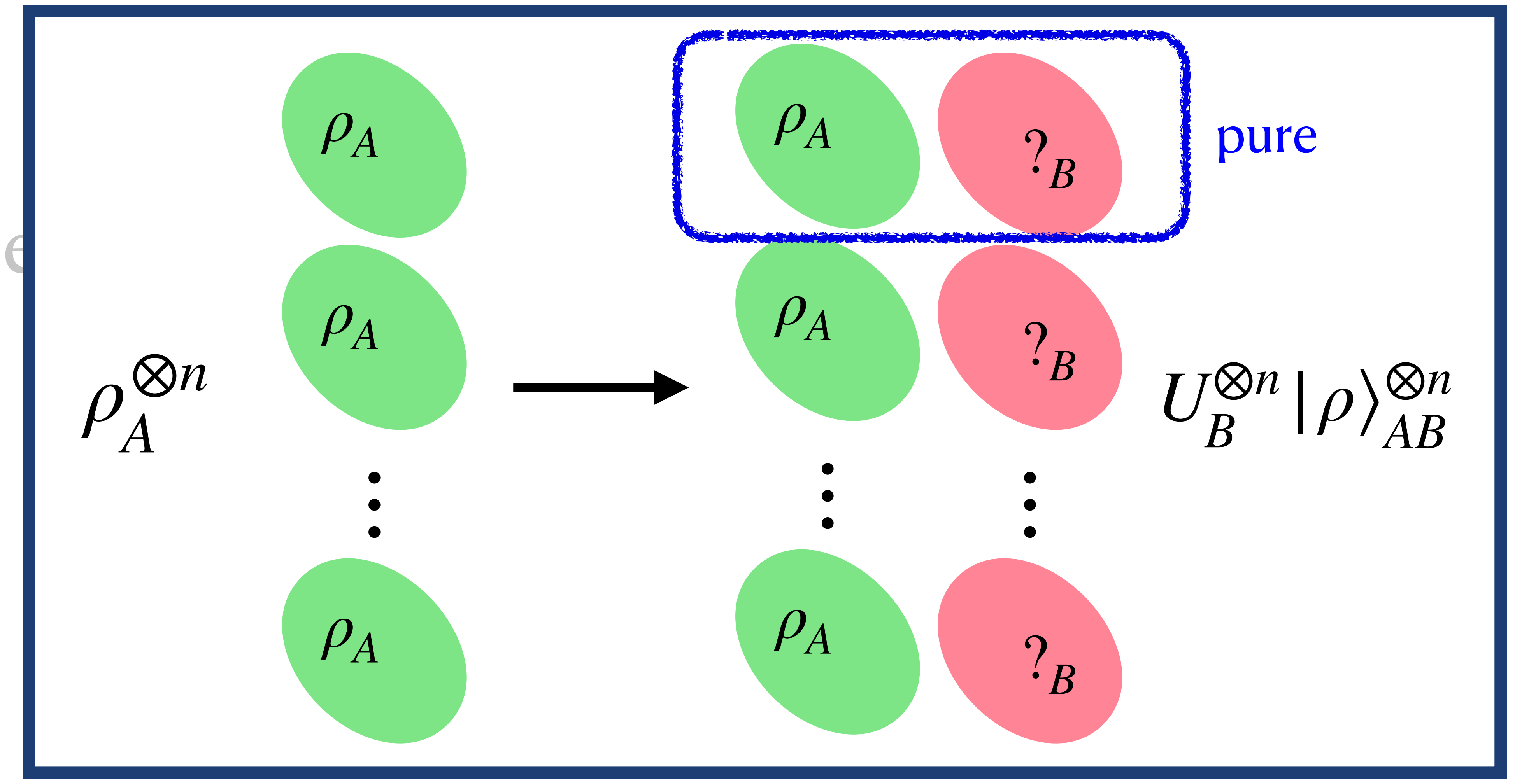
acts

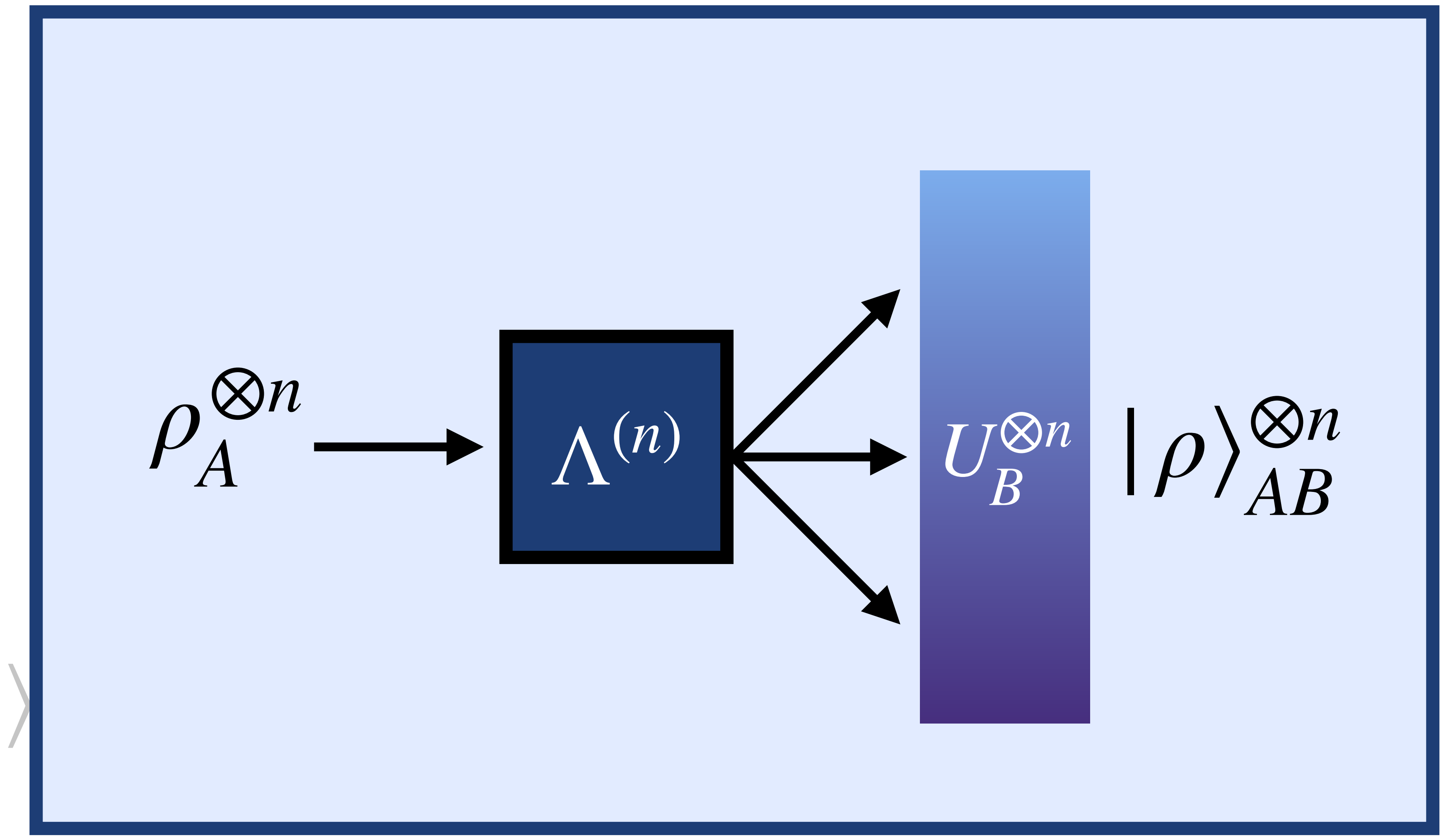
$$\tau = \frac{1}{3}|\tau_1\rangle\langle\tau_1| + \frac{1}{3}|\tau_2\rangle\langle\tau_2| + \frac{1}{3}|\tau_3\rangle\langle\tau_3|$$



$$\Lambda^{(n)}\left(\rho_A^{\otimes n}\right) = \mathbb{E}_{U_B}\Big[(\mathbb{I}_{A^n} \otimes U_B^{\otimes n})\ |\rho\rangle\langle\rho|_{AB}^{\otimes n}\ (\mathbb{I}_{A^n} \otimes U_B^{\dagger\ \otimes n})\Big],$$

where the expectation is taken over Haar-random unitaries U_B, and $|\rho\rangle$





FG, F. A. Mele, L. Lami. *Random purification channel made simple*. arXiv:2511.23451 (2025)

Simple proof

Let $\mathscr{H}_A \simeq \mathscr{H}_B$ be two isomorphic Hilbert spaces.

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$$\Gamma_{AB} := |\Gamma\rangle\langle\Gamma|_{AB}$$

$$|\Gamma\rangle_{AB} := \sum_{i=1}^{d} |e_i\rangle_A \otimes |e_i\rangle_B$$

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$$|\Gamma\rangle_{AB} := \sum_{i=1}^{d} |e_i\rangle_A \otimes |e_i\rangle_B$$

Lemma. Given $\rho_A \in \mathscr{D}(\mathscr{H}_A)$

$$[\rho_A^{\otimes n} \otimes \mathbb{I}_{B^n}, R_n] = 0$$

Standard purification. Given $\rho_A \in \mathscr{D}(\mathscr{H}_A)$

$$|\rho^{\mathrm{std}}\rangle_{AB} := \left(\sqrt{\rho_A} \otimes \mathbb{I}_B\right)|\Gamma\rangle_{AB}$$

Simple proof

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The channel

$$\Lambda^{(n)}(\,\cdot\,) := \sqrt{R_n}\left(\,\cdot \otimes \mathbb{I}_{B^n}\right)\sqrt{R_n}$$

$$\begin{aligned}
\Lambda^{(n)}\left(\rho_A^{\otimes n}\right) &= \sqrt{R_n}\,\rho_A^{\otimes n}\sqrt{R_n} = \sqrt{R_n}\sqrt{\rho_A^{\otimes n}}\sqrt{\rho_A^{\otimes n}}\sqrt{R_n} \\
&= \sqrt{\rho_A^{\otimes n}}\sqrt{R_n}\sqrt{R_n}\sqrt{\rho_A^{\otimes n}} = \sqrt{\rho_A^{\otimes n}}R_n\sqrt{\rho_A^{\otimes n}} \\
&= \mathbb{E}_{U_B}\left[(\mathbb{I}_{A^n} \otimes U_B^{\otimes n})\left(\sqrt{\rho_A}\Gamma_{AB}\sqrt{\rho_A}\right)^{\otimes n}(\mathbb{I}_{A^n} \otimes U_B^{\dagger\,\otimes n})\right] \\
&= \mathbb{E}_{U_B}\left[(\mathbb{I}_{A^n} \otimes U_B^{\otimes n})\,|\rho^{\mathrm{std}}\rangle\langle\rho^{\mathrm{std}}|_{AB}^{\otimes n}\,(\mathbb{I}_{A^n} \otimes U_B^{\dagger\,\otimes n})\right].
\end{aligned}$$

Lemma. Given $\rho_A \in \mathscr{D}(\mathscr{H}_A)$

$$[\rho_A^{\otimes n} \otimes \mathbb{I}_{B^n}, R_n] = 0$$

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$$[\rho_A^{\otimes n} \otimes \mathbb{1}_{B^n}, R_n] = 0$$

STEP 1. $[R_n, U_A^{\otimes n} \otimes \mathbb{1}_{B^n}] = 0$

Proof. $\left(U_A^{\otimes n} \otimes \mathbb{1}_{B^n}\right) R_n \left(U_A^{\otimes n} \otimes \mathbb{1}_{B^n}\right)^{\dagger} = \mathbb{E}_{W_B}\left[\left((U_A \otimes W_B)\,\Gamma_{AB}\,(U_A \otimes W_B)^{\dagger}\right)^{\otimes n}\right]$

$$= \mathbb{E}_{W_B}\left[\left(\left(\mathbb{1}_A \otimes W_B U_A^{\intercal}\right)\Gamma_{AB}\left(\mathbb{1}_A \otimes W_B U_A^{\intercal}\right)^{\dagger}\right)^{\otimes n}\right] = \mathbb{E}_{W'_B}\left[\left((\mathbb{1}_A \otimes W'_B)\,\Gamma_{AB}\left(\mathbb{1}_A \otimes W'_B\right)^{\dagger}\right)^{\otimes n}\right] = R_n\,,$$

Simple proof

Let $\mathscr{H}_A \simeq \mathscr{H}_B$ be two isomorphic Hilbert spaces.

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Lemma. Given $\rho_A \in \mathscr{D}(\mathscr{H}_A)$

$$[\rho_A^{\otimes n} \otimes \mathbb{I}_{B^n}, R_n] = 0$$

STEP 1. $[R_n, U_A^{\otimes n} \otimes \mathbb{I}_{B^n}] = 0$

STEP 2. $\left[R_n, \left(\sum_{j=1}^{n} H^{(j)}\right) \otimes \mathbb{I}_{B^n}\right] = 0$ where $H^{(j)} := \mathbb{I}^{\otimes(j-1)} \otimes H \otimes \mathbb{I}^{\otimes(n-j)}$, with H arbitrary Hermitian operator

Proof. $\left[R_n, \left(e^{itH}\right)^{\otimes n} \otimes \mathbb{I}_{B^n}\right] = 0, \quad t \in \mathbb{R} \quad \rightarrow \quad \left.\frac{d}{dt}\right|_{t=0} \quad \rightarrow \quad \left[R_n, \left(\sum_{j=1}^{n} H^{(j)}\right) \otimes \mathbb{I}_{B^n}\right] = 0$

Simple proof

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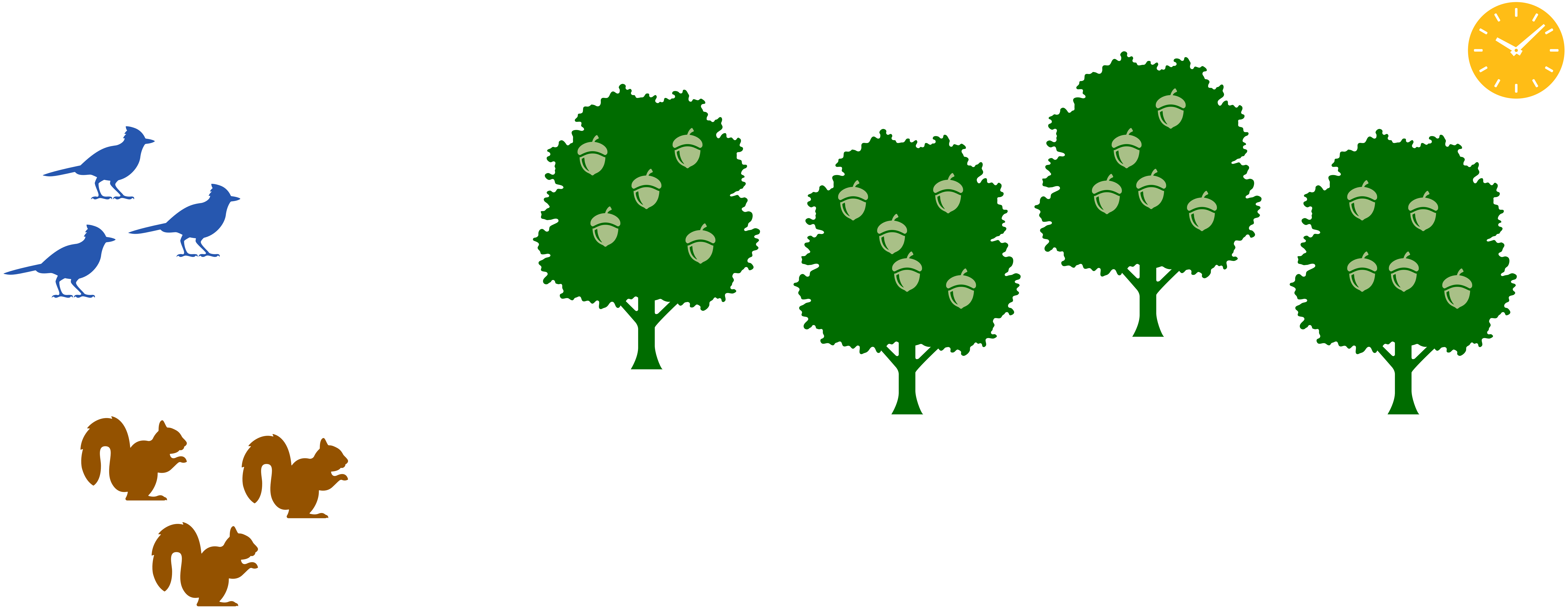
STEP 2. $\left[R_n, \left(\sum_{j=1}^n H^{(j)}\right) \otimes \mathbb{I}_{B^n}\right] = 0$ where $H^{(j)} := \mathbb{I}^{\otimes(j-1)} \otimes H \otimes \mathbb{I}^{\otimes(n-j)}$, with H arbitrary Hermitian operator

STEP 3. $\left[R_n, \rho_A^{\otimes n} \otimes \mathbb{I}_{B^n}\right] = 0$

Proof. $0 = \left[R_n, \exp\left(\sum_{j=1}^n H^{(j)}\right) \otimes \mathbb{I}_{B^n}\right] = \left[R_n, \left(e^H\right)^{\otimes n} \otimes \mathbb{I}_{B^n}\right]$

$\rho_A > 0 \quad \rightarrow \quad \rho_A = \dfrac{e^H}{\operatorname{Tr} e^H} \quad \rightarrow \quad \left[R_n, \rho_A^{\otimes n} \otimes \mathbb{I}_{B^n}\right] = 0$ (extension to $\rho_A \geq 0$ by continuity)

“Acorn trick”



E. Tang, J. Wright, and M. Zhandry. *Conjugate queries can help*. Preprint arXiv:2510.07622 (2025)

“Acorn trick”

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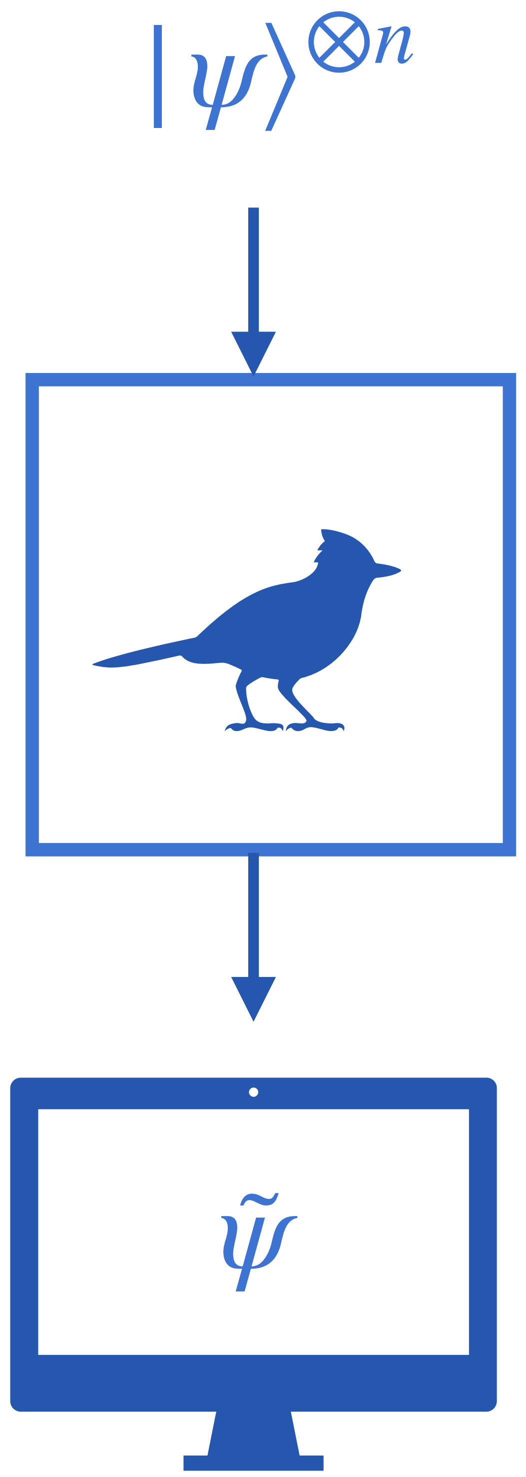
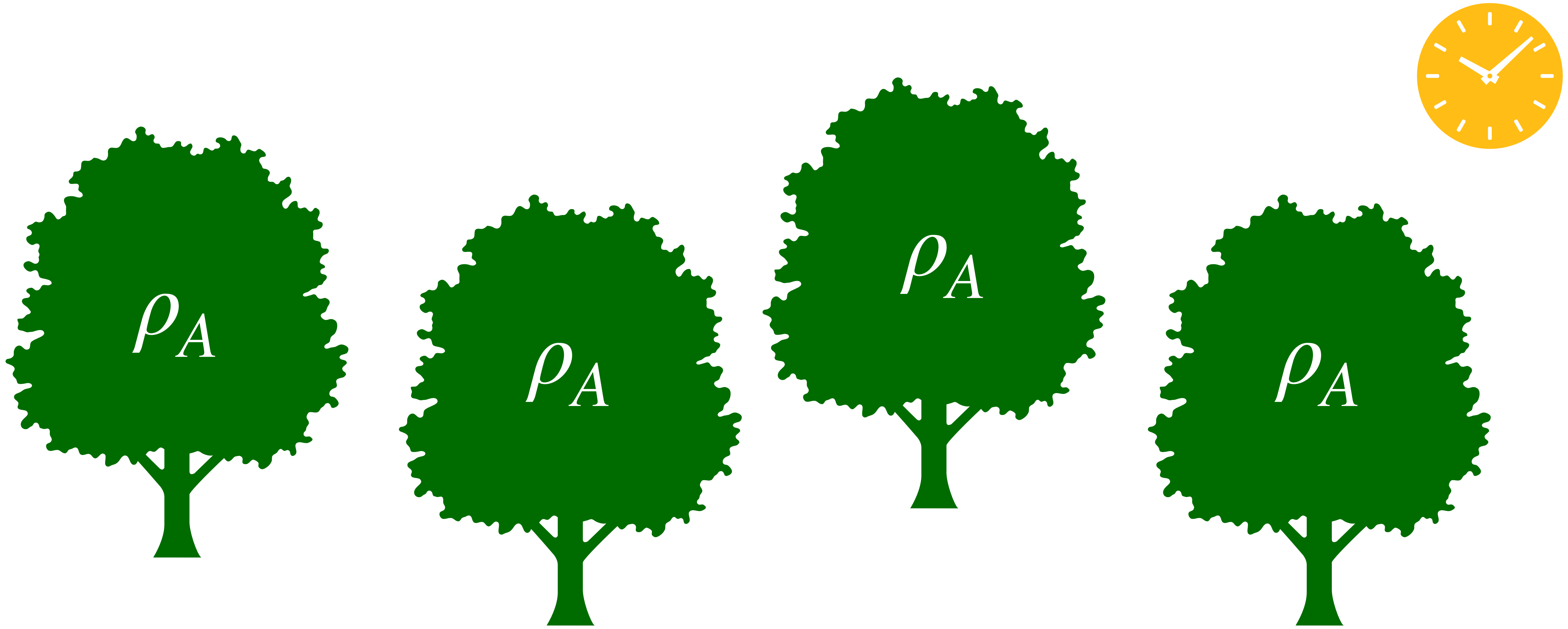
Applications in learning theory



n i.i.d. copies
of an unknown pure state

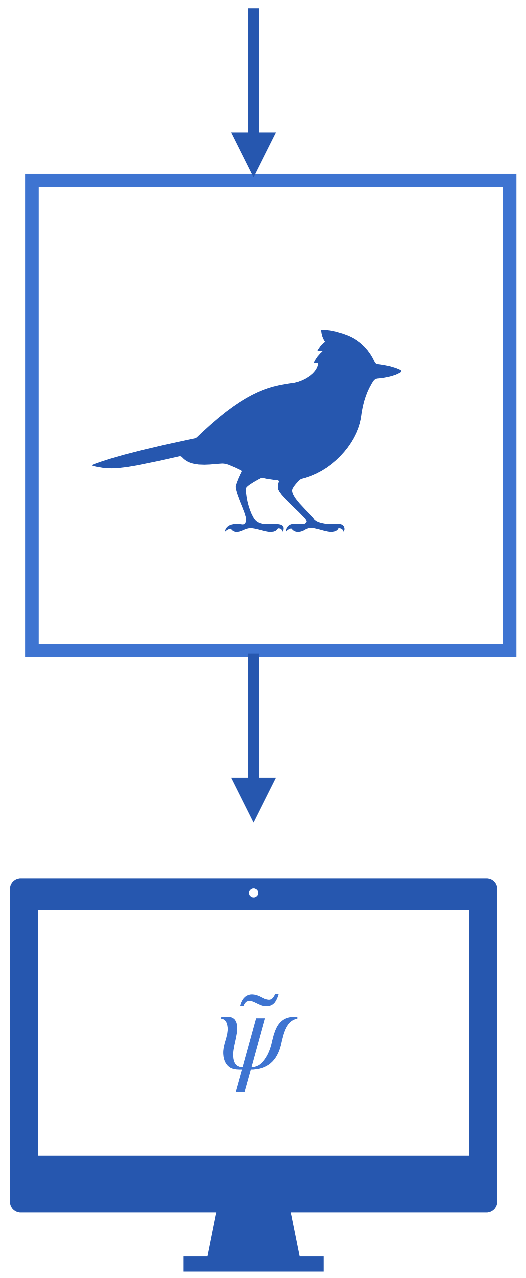
efficient learning
algorithm for pure states

classical description
of the unknown state



Applications in learning theory

$|\psi\rangle^{\otimes n}$



n i.i.d. copies
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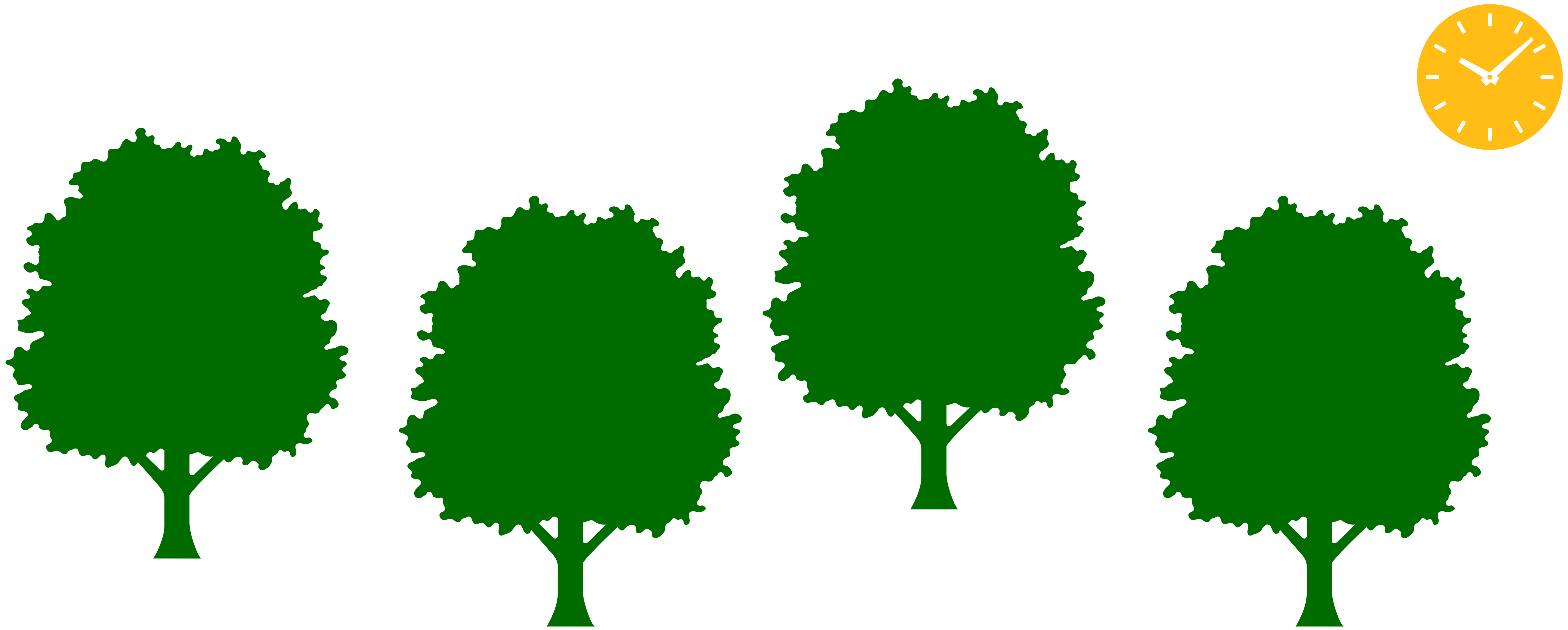
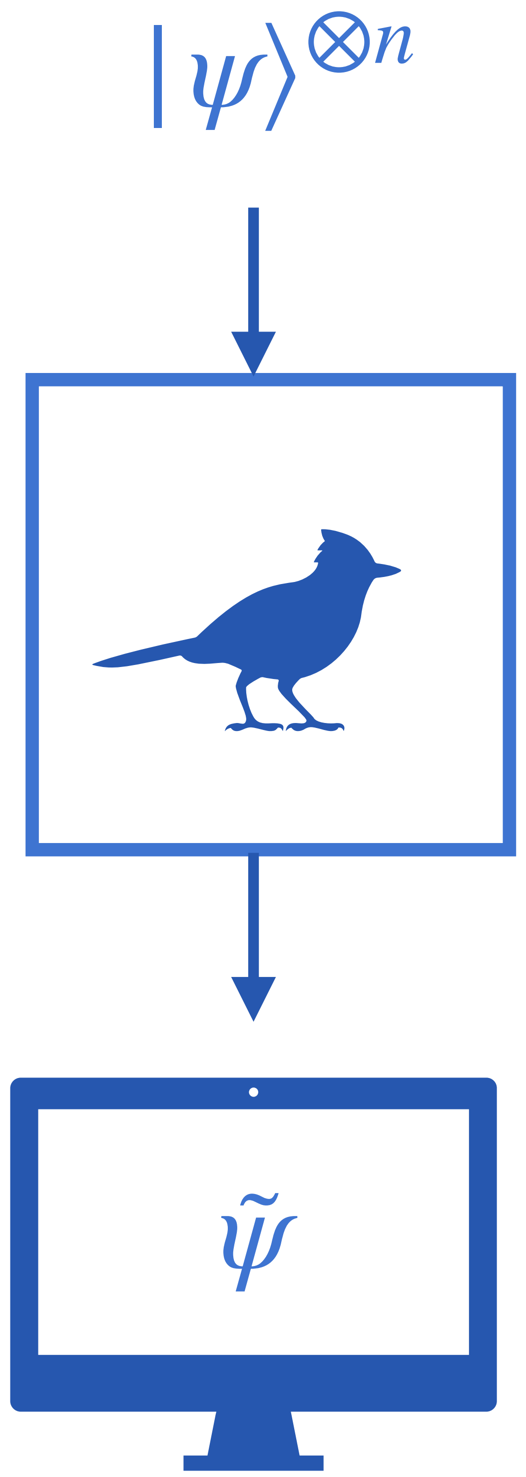


$$|\rho^{(1)}\rangle_{AB} \otimes |\rho^{(2)}\rangle_{AB} \otimes |\rho^{(3)}\rangle_{AB} \otimes |\rho^{(4)}\rangle_{AB}$$

Applications in learning theory



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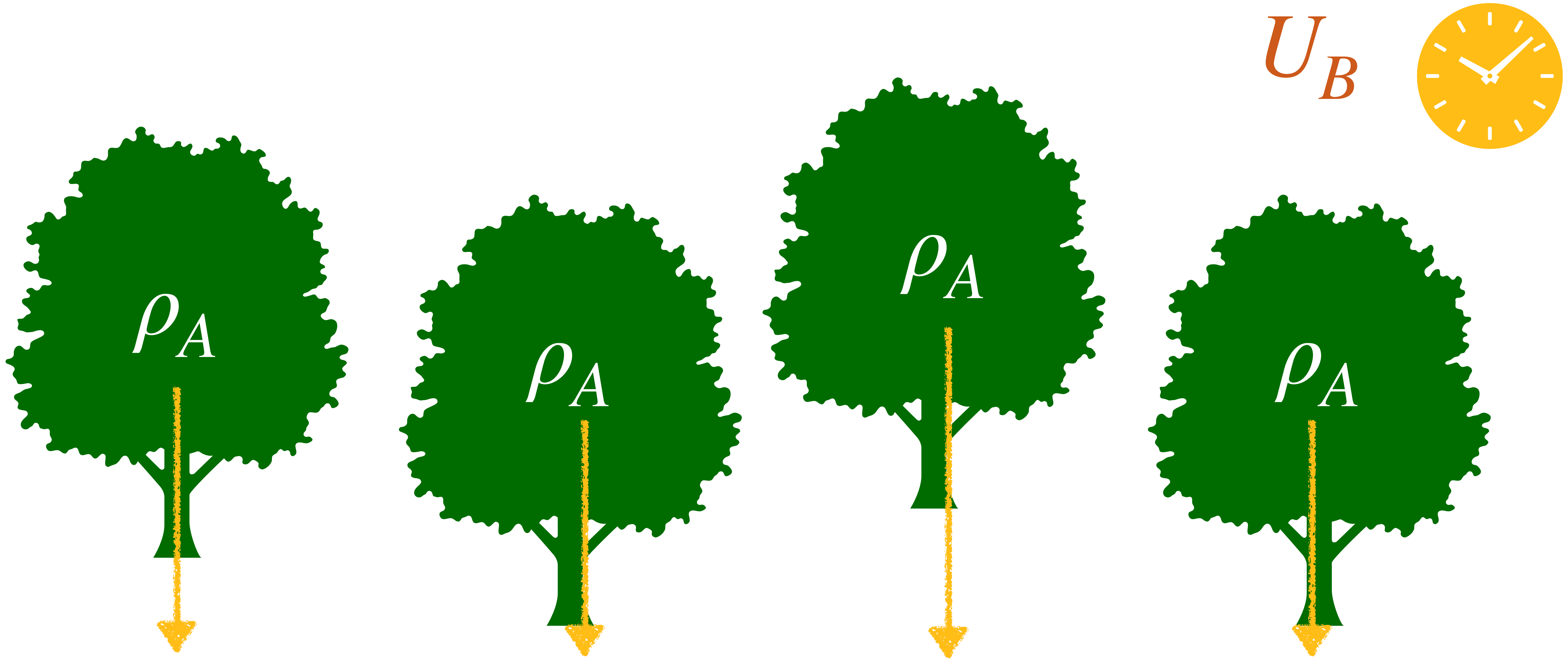
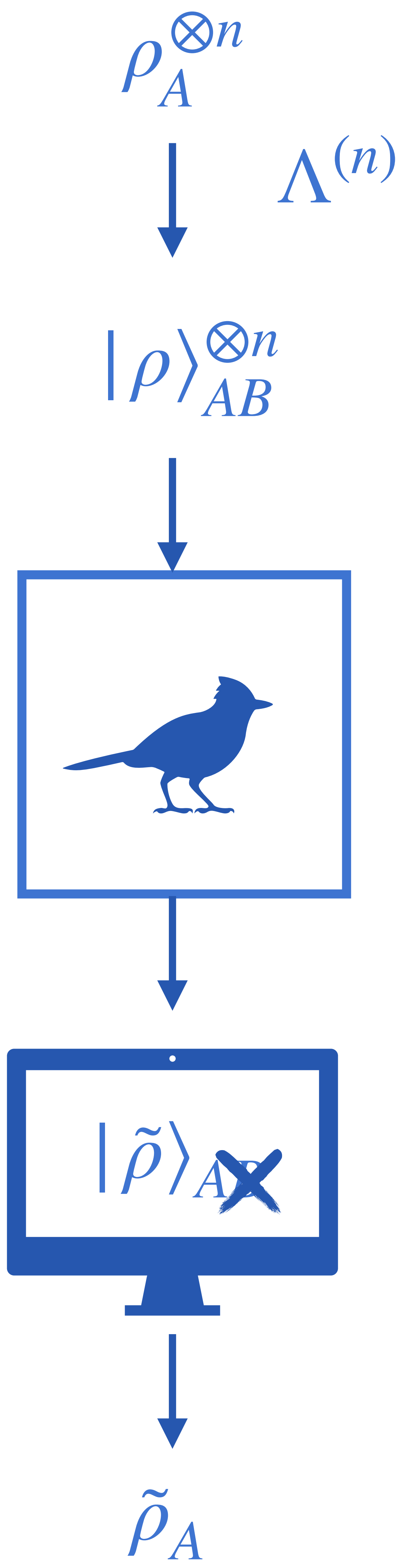


$$U_B|\rho^{\mathrm{std}}\rangle_{AB} \otimes U_B|\rho^{\mathrm{std}}\rangle_{AB} \otimes U_B|\rho^{\mathrm{std}}\rangle_{AB} \otimes U_B|\rho^{\mathrm{std}}\rangle_{AB}$$

$$\boxed{\Lambda^{(n)}\left(\rho_A^{\otimes n}\right) = \mathbb{E}_{U_B}\left[\left(\mathbb{I}_{A^n} \otimes U_B^{\otimes n}\right) |\rho\rangle\langle\rho|_{AB}^{\otimes n} \left(\mathbb{I}_{A^n} \otimes U_B^{\dagger\,\otimes n}\right)\right]}$$

Applications in learning theory



1. State tomography

A. Pelecanos, J. Spilecki, E. Tang, and J. Wright.
Mixed state tomography reduces to pure state tomography. arXiv:2511.15806 (2025)

$$N = O\left(\frac{rd + \log(1/\delta)}{\varepsilon}\right)$$

2. Channel tomography

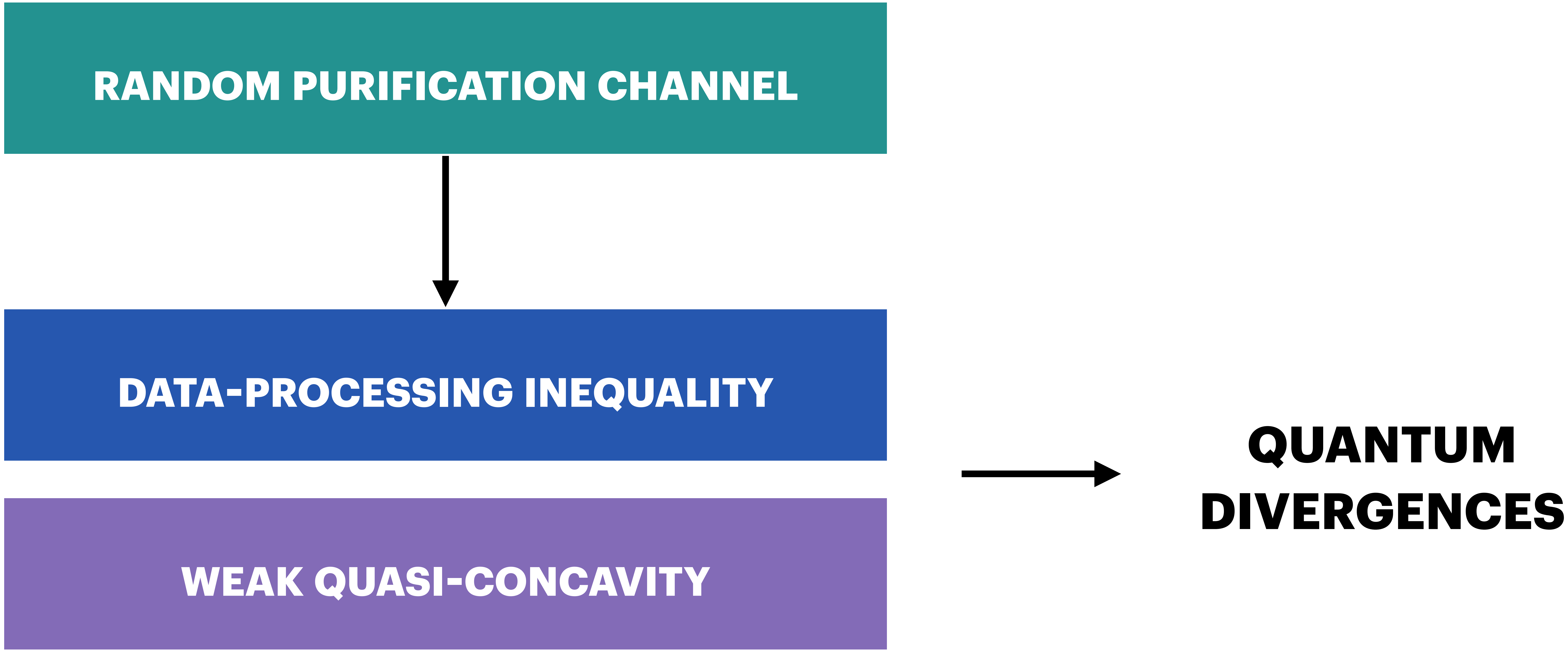
A. A. Mele, L. Bittel
Optimal learning of quantum channels in diamond distance. arXiv:2512.10214 (2025)

$$N = O\left(\frac{d_{\text{in}} d_{\text{out}} r}{\varepsilon^2}\right)$$

3. Fidelity estimation

T. Utsumi, Y. Nakata, Q. Wang, and R. Takagi.
Quantum algorithms for Uhlmann transformation. arXiv:2509.03619 (2025)

Applications in quantum Shannon theory



RANDOM PURIFICATION CHANNEL

DATA-PROCESSING INEQUALITY

WEAK QUASI-CONCAVITY

QUANTUM DIVERGENCES

Weak quasi-concavity

Quantum divergence $\mathbb{D}(\rho\|\sigma) \geq \mathbb{D}(\Lambda(\rho)\|\Lambda(\sigma))$ Λ quantum channel

Weak quasi-concavity $\mathbb{D}\left(\sum_{i=1}^{N} p_i\rho_i \,\middle\|\, \sigma \right)$

Weak quasi-concavity

Quantum divergence $\mathbb{D}(\rho\|\sigma) \geq \mathbb{D}(\Lambda(\rho)\|\Lambda(\sigma))$ Λ quantum channel

Weak quasi-concavity $\mathbb{D}\left(\xcancel{\sum_{i=1}^{N}} p_i\rho_i \,\middle\|\, \sigma\right) \geq \sum_{i=1}^{N} p_i D(\rho_i\|\sigma)$

Weak quasi-concavity

Quantum divergence $\mathbb{D}(\rho\|\sigma) \geq \mathbb{D}(\Lambda(\rho)\|\Lambda(\sigma))$ Λ quantum channel

Weak quasi-concavity $\mathbb{D}\left(\xcancel{\sum_{i=1}^{N}} p_i\rho_i \,\middle\|\, \sigma\right) \geq \min_{1\leq i\leq N} \mathbb{D}(\rho_i\|\sigma) - \log \mathrm{poly}_d(N, s_\sigma)$

for any finite ensemble of states $\{(p_i,\rho_i)\}_{i=1,\ldots,N}$ on $\mathscr{H}^{\otimes n}$ and any $\sigma \in \mathscr{D}(\mathscr{H}^{\otimes n})$

Weak quasi-concavity

Quantum divergence $\mathbb{D}(\rho\|\sigma) \geq \mathbb{D}(\Lambda(\rho)\|\Lambda(\sigma))$ Λ quantum channel

Weak quasi-concavity $\mathbb{D}\left(\cancel{\sum_{i=1}^{N}} p_i\rho_i \,\middle\|\, \sigma\right) \geq \min_{1\leq i\leq N} \mathbb{D}(\rho_i\|\sigma) - \log \mathrm{poly}_d(N, s_\sigma)$

for any finite ensemble of states $\{(p_i, \rho_i)\}_{i=1,\ldots,N}$ on $\mathscr{H}^{\otimes n}$ and any $\sigma \in \mathscr{D}(\mathscr{H}^{\otimes n})$

Quantum relative entropy $D(\rho\|\sigma) := \mathrm{Tr}[\rho(\log\rho - \log\sigma)]$

Weak quasi-concavity $D\left(\sum_i p_i\rho_i \,\middle\|\, \sigma\right) \geq \sum_i p_i D(\rho_i\|\sigma) + \sum_i p_i \log p_i$

- sandwiched Rényi divergences $\tilde{D}_\alpha$ of order $\alpha \in (0,\infty]$
- measured Rényi divergences $D_{M,\alpha}$ of order $\alpha \in (0,\infty]$
- #-Rényi divergences $D_\alpha^\#$ or order $\alpha \in (1,\infty]$

Applications in quantum Shannon theory

Uhlmann's theorem for quantum divergences

Fidelity: $F(\rho, \sigma) := \left\| \sqrt{\rho}\sqrt{\sigma} \right\|_1^2$

For pure states $F\big(\,|\psi\rangle, |\phi\rangle\big) = |\langle \psi | \phi \rangle|^2$

$$\boxed{F(\rho, \sigma) = \max_{|\sigma\rangle} |\langle \rho | \sigma \rangle|^2}$$

Uhlmann's theorem

$$\boxed{\mathbb{D}^{\infty}(\rho_A \| \sigma_A) = \mathbb{D}^{\infty}\left(\rho_{AB} \| \mathscr{C}_{AB}^{\sigma_A}\right)}$$

Uhlmann's theorem for divergences

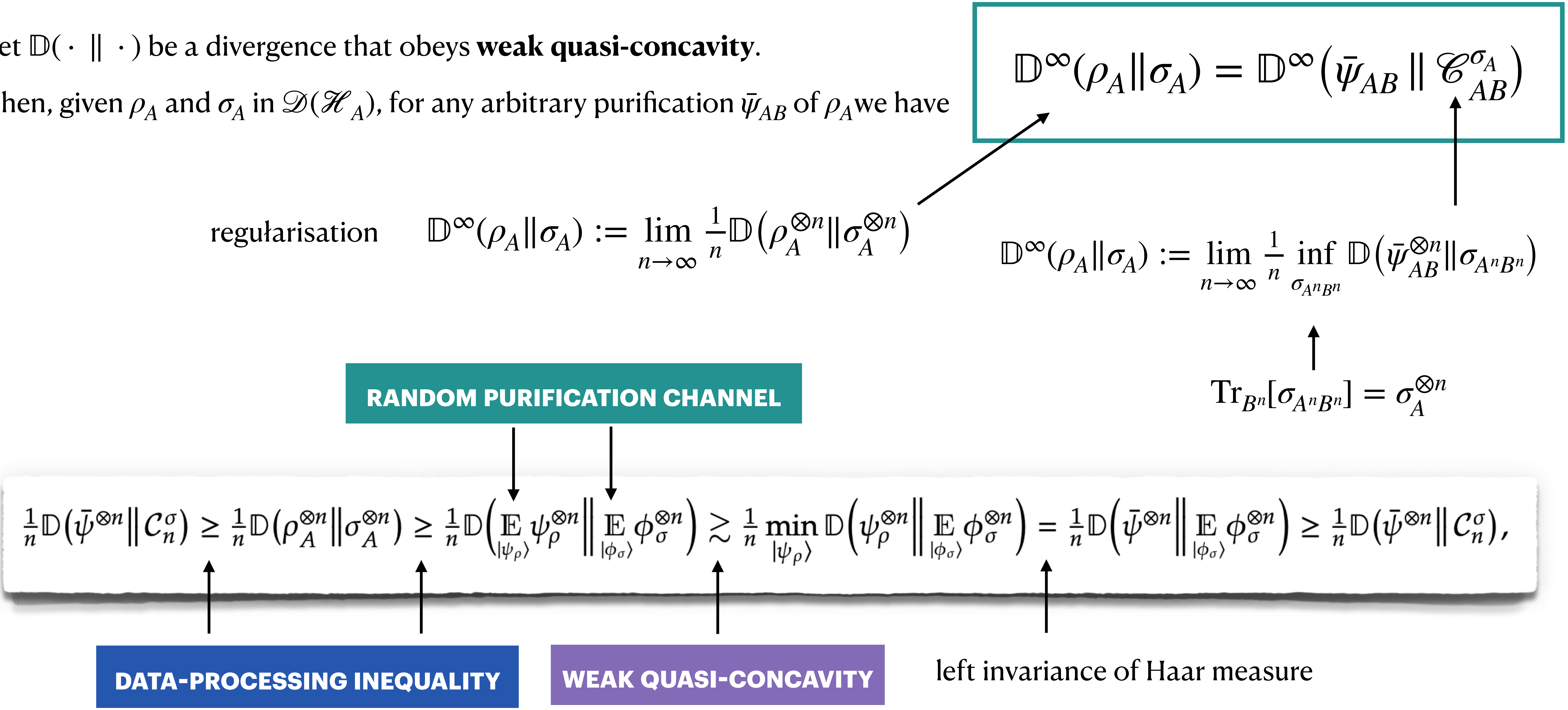
G. Mazzola, D. Sutter, and R. Renner, *Uhlmann's theorem for relative entropies*. IEEE Trans. Inf. Theory, 71(9):7039–7051 (2025)
K. Fang, H. Fawzi, and O. Fawzi, *Variational expressions and Uhlmann theorem for measured divergences*. IEEE Trans. Inf. Theory, 72(3):1751-1760 (2026)

Uhlmann’s theorem for divergences

in one line

Let $\mathbb{D}(\cdot\,\|\,\cdot)$ be a divergence that obeys **weak quasi-concavity**.

Then, given ρ_A and σ_A in $\mathscr{D}(\mathscr{H}_A)$, for any arbitrary purification $\bar{\psi}_{AB}$ of $\rho_A$we have



$$\boxed{\mathbb{D}^\infty(\rho_A\|\sigma_A) = \mathbb{D}^\infty\left(\bar{\psi}_{AB}\,\|\,\mathscr{C}^{\sigma_A}_{AB}\right)}$$

regularisation $\mathbb{D}^\infty(\rho_A\|\sigma_A) := \lim_{n\to\infty}\frac{1}{n}\mathbb{D}\left(\rho_A^{\otimes n}\|\sigma_A^{\otimes n}\right)$

$$\mathbb{D}^\infty(\rho_A\|\sigma_A) := \lim_{n\to\infty}\frac{1}{n}\inf_{\sigma_{A^nB^n}}\mathbb{D}\left(\bar{\psi}_{AB}^{\otimes n}\|\sigma_{A^nB^n}\right)$$

$$\mathrm{Tr}_{B^n}[\sigma_{A^nB^n}] = \sigma_A^{\otimes n}$$

RANDOM PURIFICATION CHANNEL

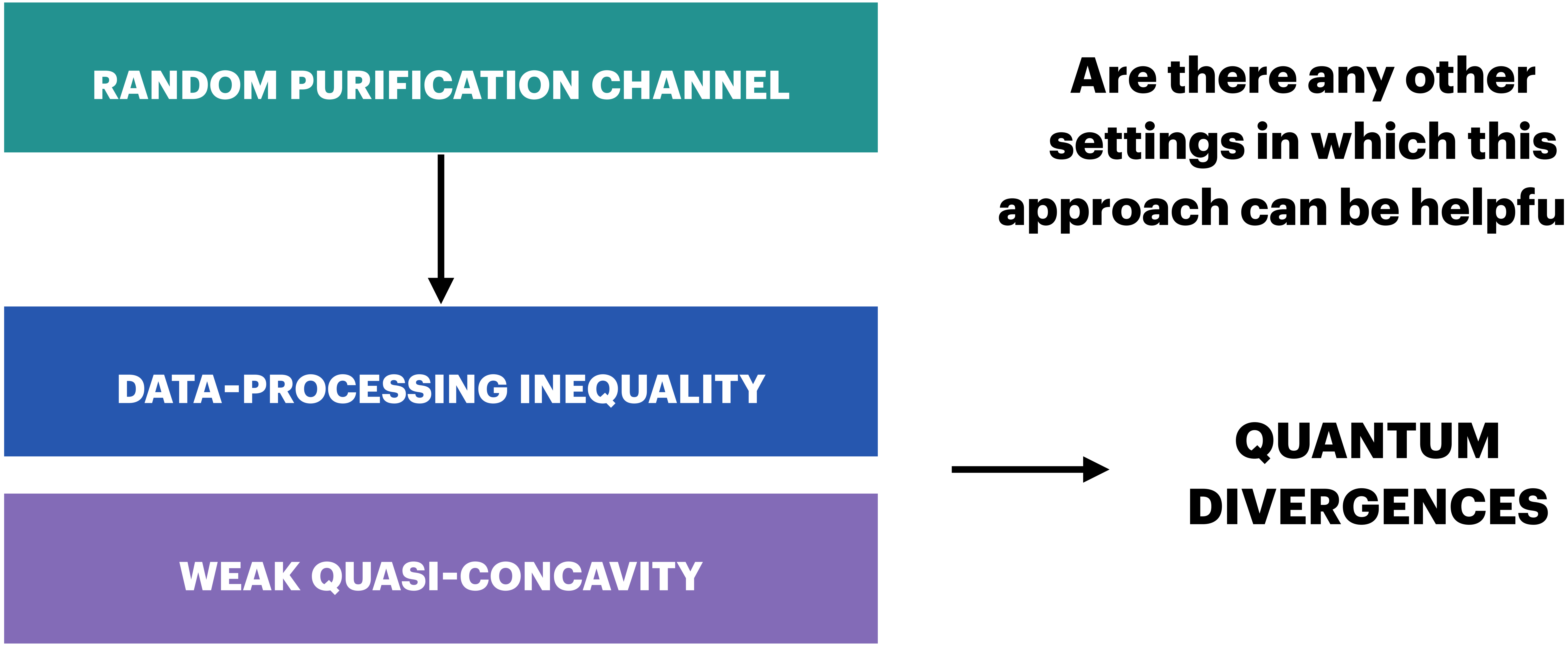
$$\frac{1}{n}\mathbb{D}\left(\bar{\psi}^{\otimes n}\middle\|\mathcal{C}^\sigma_n\right) \geq \frac{1}{n}\mathbb{D}\left(\rho_A^{\otimes n}\middle\|\sigma_A^{\otimes n}\right) \geq \frac{1}{n}\mathbb{D}\left(\mathop{\mathbb{E}}_{|\psi_\rho\rangle}\psi_\rho^{\otimes n}\middle\|\mathop{\mathbb{E}}_{|\phi_\sigma\rangle}\phi_\sigma^{\otimes n}\right) \gtrsim \frac{1}{n}\min_{|\psi_\rho\rangle}\mathbb{D}\left(\psi_\rho^{\otimes n}\middle\|\mathop{\mathbb{E}}_{|\phi_\sigma\rangle}\phi_\sigma^{\otimes n}\right) = \frac{1}{n}\mathbb{D}\left(\bar{\psi}^{\otimes n}\middle\|\mathop{\mathbb{E}}_{|\phi_\sigma\rangle}\phi_\sigma^{\otimes n}\right) \geq \frac{1}{n}\mathbb{D}\left(\bar{\psi}^{\otimes n}\middle\|\mathcal{C}^\sigma_n\right),$$

DATA-PROCESSING INEQUALITY

WEAK QUASI-CONCAVITY

left invariance of Haar measure

Applications in quantum Shannon theory



Are there any other settings in which this approach can be helpful?

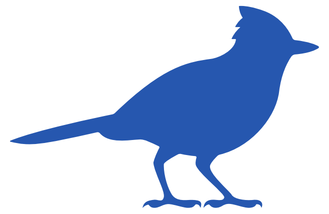
Random purification channel BEYOND IID

Permutation symmetry

$$\rho_{A^n} = U_\pi \rho_{A^n} U_\pi^\dagger \qquad \forall \pi \in S_n$$

$$\boxed{\Lambda^{(n)}\left(\rho_{A^n}\right) = \mathbb{E}_{U_B}\left[(\mathbb{I}_{A^n} \otimes U_B^{\otimes n})\, |\rho\rangle\langle\rho|_{A^nB^n}\, (\mathbb{I}_{A^n} \otimes {U_B^\dagger}^{\otimes n})\right]}$$

Thank you for your attention!



FG, F. A. Mele, L. Lami. *Random purification channel made simple*. arXiv:2511.23451 (2025)
F. A. Mele, FG, S. Chen, M. Fanizza, L. Lami. *Random purification channel for passive Gaussian bosons*. arXiv:2512.16878 (2025)
M. Walter, F. Witteveen. *A random purification channel for arbitrary symmetries with applications to fermions and bosons*. arXiv:2512.15690 (2025)