Subsystem Quantum Error Correction for Noisy Quantum Metrology

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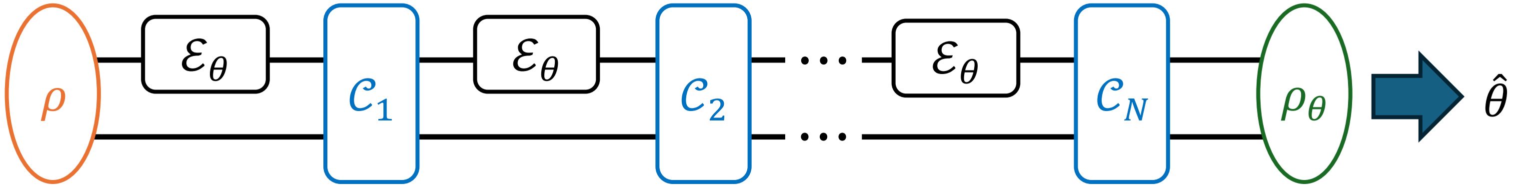
Outline

- Standard error correction for metrology
- Subsystem stabilizer codes
- Conditions for Heisenberg limit using subsystem codes
- Dynamical codes and application to metrology of time-varying signals

Quantum metrology: estimating unknown process

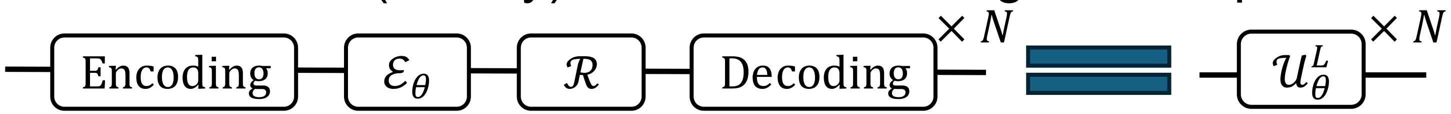
- Given N queries to a quantum channel $\mathcal{E}_\theta$, estimate θ



- Minimize estimation precision $\delta\hat{\theta}^2 = \mathbb{E}\left[(\hat{\theta}-\theta)^2\right]$
- $\delta\hat{\theta}^2 \sim 1/F^Q(\rho_\theta)$ quantum Fisher information (QFI)
- Heisenberg limit (HL): $\delta\hat{\theta}^2 \sim 1/N^2$ vulnerable to noise
- Standard quantum limit (SQL): $\delta\hat{\theta}^2 \sim 1/N$ limit of classical resources (without entanglement or coherence)

Quantum error correction for metrology

- $\mathcal{E}_\theta(\rho) = \mathcal{N} \circ \mathcal{U}_\theta(\rho) = \sum_i K_i^\theta \rho K_i^{\theta\dagger}$ where $\mathcal{U}_\theta(\rho) = e^{-i\theta H} \rho e^{i\theta H}$
- Correct errors, preserve signal
- QEC constructs a noiseless (unitary) evolution in the logical subspace



- Probe state $|\psi_0\rangle = \frac{1}{\sqrt{2}}(|0_L\rangle + |1_L\rangle)$, Hamiltonian $H = Z_L$,

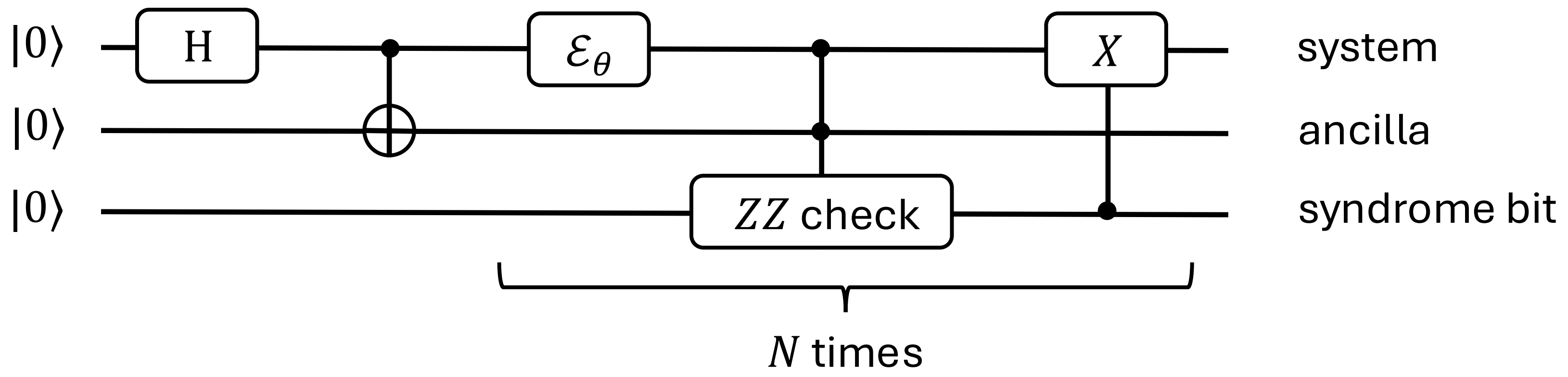
then output state $|\psi_\theta\rangle = \frac{1}{\sqrt{2}}(|0_L\rangle + e^{2Ni\theta}|1_L\rangle)$, QFI $F^Q(\rho_\theta) \propto N^2$

- Achievability of HL by a specific QEC code C $\Longleftrightarrow$ $\Pi_C K_\alpha^{\theta\dagger} K_\beta^\theta \Pi_C \propto \Pi_C$, and $\Pi_C H \Pi_C \not\propto \Pi_C$
- Achievability of HL by QEC $\Longleftrightarrow$ $H \notin \mathrm{Span}\{K_\alpha^{\theta\dagger} K_\beta^\theta, \forall \alpha, \beta\}$ (HNKS condition)

Demkowicz-Dobrzański, Czajkowski, Sekatski, PRX, 7, 041009 (2017)
Zhou, Zhang, Preskill, Jiang, Nature Communications, 9, 78 (2018)
Zhou, Jiang, PRX Quantum, 2, 010343 (2021)

Example: phase estimation under bit flip noise

- $\mathcal{E}_\theta(\rho) = \mathcal{N} \circ \mathcal{U}_\theta(\rho)$
  - $\mathcal{U}_\theta(\rho) = e^{-\mathrm{i}\theta H} \rho e^{\mathrm{i}\theta H}$ for $H = Z$
  - $\mathcal{N}(\rho) = \sum_i E_i \rho E_i^\dagger$ for $E_1 = \sqrt{1-p} I, E_2 = \sqrt{p} X$
- One noiseless ancilla qubit is needed for code construction
- Prepare probe state $|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$, which is stabilized by $Z_1 Z_2$
- Syndrome measurement projections: $P_1 = |00\rangle\langle 00| + |11\rangle\langle 11|, P_2 = |01\rangle\langle 01| + |10\rangle\langle 10|$
- Recovery: $R_1 = U_1 P_1, R_2 = U_2 P_2$ for $U_1 = I$ and $U_2 = X_S \otimes I_A$



Towards experiment-friendly QEC via subsystem codes

- Can we reduce or even eliminate the syndrome measurement overhead?
- Can we use fewer or even no ancilla qubits to implement QEC?
- Subsystem codes provide a promising solution 😊
- Idea: one may protect the logical information of interest in a "subsystem", rather than the whole code subspace
- Standard QEC: $\mathcal{H} = \mathcal{H}_C \oplus \mathcal{H}_C^{\perp}$
- Subsystem QEC: $\mathcal{H}_C = \underbrace{\mathcal{H}_L}_{\text{logical}} \otimes \underbrace{\mathcal{H}_G}_{\text{gauge}}$
- Measuring gauge (check measurements) may concern lower-weight Pauli operators
- If we can "separate" the signal and noise via tensor product, no syndrome measurements are required

Kribs, Laflamme, Poulin, PRL 94, 180501 (2005)

Stabilizer codes

- Pauli group on n qubits $\mathcal{P}_n$ (tensor product of Pauli operators with phase $\pm 1, \pm \mathrm{i}$)
- Stabilizer group: $\mathcal{S} = \langle Z'_1, \dots, Z'_s \rangle$ (set of syndromes)
- Normalizer

$$\mathcal{N}(\mathcal{S}) = \langle \mathrm{i}, \underbrace{Z'_1, \dots, Z'_s}_{s \text{ stabilizers}}, \underbrace{Z'_{s+1}, X'_{s+1}, \dots, Z'_n, X'_n}_{k \text{ logical qubits}} \rangle$$

- Logical subspace is stabilized by $\mathcal{S}$ ($+1$ eigenspace)
- Number of logical qubits $k = n - s$
- Correctable errors: $E_i E_j \notin \mathcal{N}(\mathcal{S}) - \mathcal{S}$ (no logical error)

Knill, Laflamme, PRA 55, 900 (1997)

Subsystem stabilizer codes

- Pauli group on n qubits $\mathcal{P}_n$
- Stabilizer group: $\mathcal{S} = \langle Z'_1, \dots, Z'_s \rangle$
- Normalizer $\mathcal{N}(\mathcal{S}) = \langle i, Z'_1, \dots, Z'_n, X'_{s+1}, \dots, X'_n \rangle$
- Gauge group (set of check operators)

$$\mathcal{G} = \langle i, \underbrace{Z'_1, \dots, Z'_s}_{s \text{ stabilizers}}, \underbrace{Z'_{s+1}, \dots, Z'_{s+g}, X'_{s+1}, \dots, X'_{s+g}}_{g \text{ gauge qubits}}, \underbrace{Z'_{s+g+1}, X'_{s+g+1}, \dots, Z'_n, X'_n}_{k \text{ logical qubits}} \rangle$$

- Number of logical qubits $k = n - s - g$
- Correctable errors: $E_i E_j \notin \mathcal{N}(\mathcal{S}) - \mathcal{G}$
- $(\mathcal{S}, \mathcal{G})$ defines a subsystem code

Poulin, PRL 95, 230504 (2005)

Effective dynamics via subsystem QEC

- $\rho_\theta = \mathcal{R} \circ \mathcal{E}_\theta(\rho) = \mathcal{R} \circ \mathcal{N} \circ \mathcal{U}_\theta(\rho)$ around $\theta = 0$ (first order expansion)
- $\rho = \rho_L \otimes \rho_G$ is supported on the code space $\mathcal{H}_C$

  logical gauge

- $\rho_\theta = \left(\rho_L - i[H_{\text{eff}}, \rho_L]d\theta + O(d\theta^2)\right) \otimes \rho_G$
- Factorizing projected Hamiltonian $\Pi_C H \Pi_C = \sum_k H_{L,k} \otimes H_{G,k}$
- Effective Hamiltonian $H_{\text{eff}} = \sum_k \langle H_{G,k} \rangle_{\rho_G} H_{L,k}$
- Gauge state reset is generally required in recovery operation $\mathcal{R}$ in metrology

HL via subsystem QEC

- QFI $F^Q(\rho_\theta) \sim N^2$ indicates HL, for $\rho_\theta = (\mathcal{R} \circ \mathcal{E}_\theta)^N(\rho)$
- **Theorem 1:** Consider multi-qubit Pauli noise $\{E_i\}$. The Hamiltonian $H = \sum_k \lambda_k Q_k$ is decomposed in the Pauli basis $\{Q_k\}$. A subsystem stabilizer code $(\mathcal{S}, \mathcal{G})$ achieves the HL if

1) $E_i E_j \notin \mathcal{N}(\mathcal{S}) - \mathcal{G},\ \forall i, j$ (Knill-Laflamme condition)

2) There exists at least one Pauli component $Q_{k_0} \in \mathcal{N}(\mathcal{S}) - \mathcal{G}$ and $\sum_{j: Q_j Q_{k_0} \in \mathcal{S}} \lambda_j \neq 0$ (Net contribution to H_{eff} exists)

Syndrome-free HL

- **Theorem 2:** When $H \notin \mathrm{Span}\{\langle E_1, \dots, E_r \rangle\}$, the HL can be achieved by a syndrome-free subsystem stabilizer code $(\{I\}, \mathcal{G})$ with at most one noiseless ancilla qubit. This ancilla can be omitted in certain cases.
- Proof sketch:
- $H = \sum_k \lambda_k Q_k$ contains a Pauli component Q_{k_0} outside the noise algebra.
- By a Clifford transformation U,

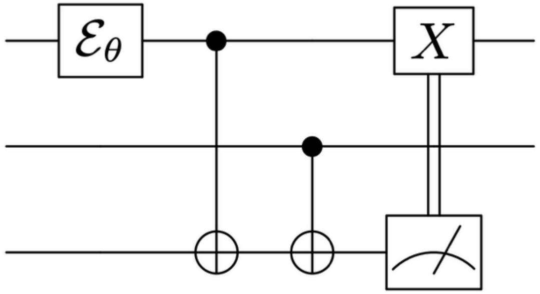
$$\langle E_1, \dots, E_r \rangle / \{\pm 1, \pm \mathrm{i}\} \cong \langle Z_1, \dots, Z_\alpha, X_1, \dots, X_\beta \rangle$$

$$Q_{k_0} \cong Z_{\alpha+1}$$

- In general $\alpha \le n$. An ancilla is required only if $\alpha = n$.

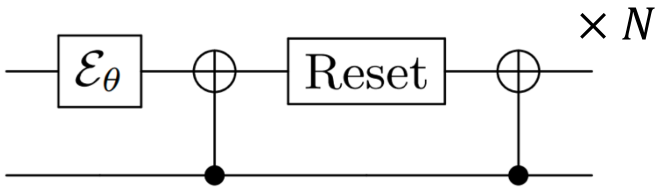
Revisiting the example

Recap: standard QEC



- $\mathcal{E}_\theta(\rho) = \mathcal{N} \circ \mathcal{U}_\theta(\rho)$
  - ➢ $\mathcal{U}_\theta(\rho) = e^{-\mathrm{i}\theta H} \rho e^{\mathrm{i}\theta H}$ for $H = Z$
  - ➢ $\mathcal{N}(\rho) = \sum_i E_i \rho E_i^\dagger$ for $E_1 = \sqrt{1-p} I, E_2 = \sqrt{p} X$
- Single-qubit case

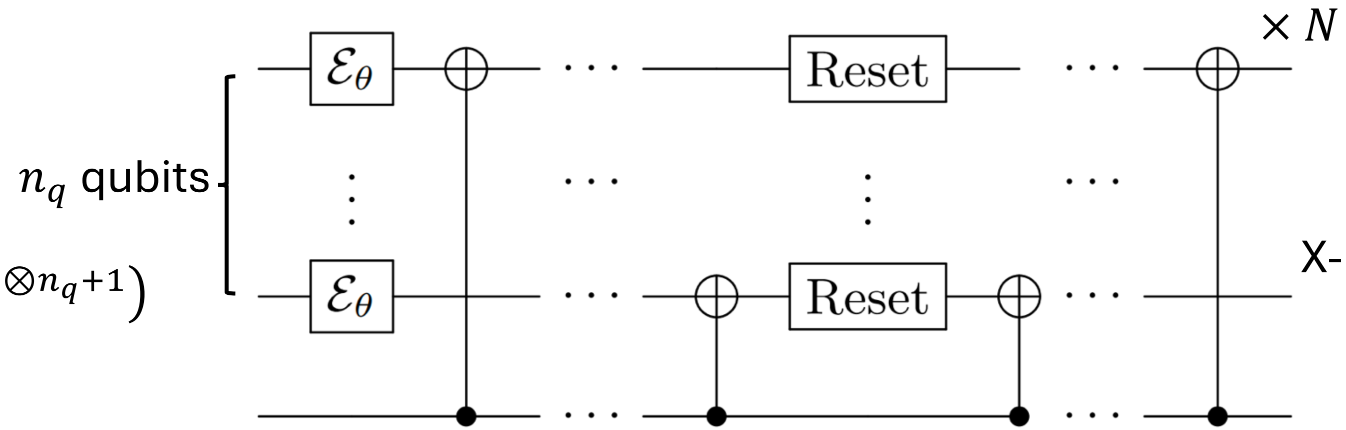
$$|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$



X-basis measurements

- Multiple-qubit case

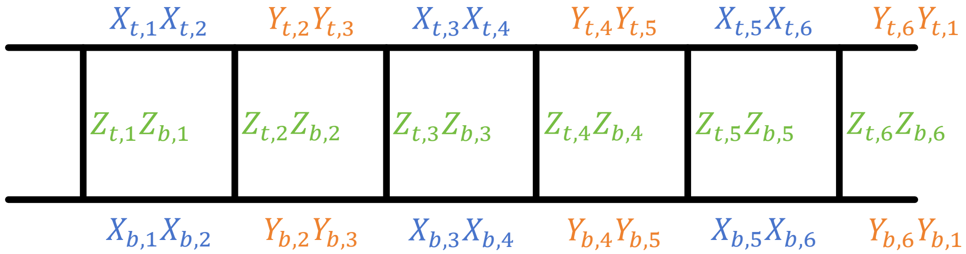
$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle^{\otimes n_q+1} + |1\rangle^{\otimes n_q+1}\right)$$



X-basis measurements

Floquet (dynamical) codes

- Dynamical logical subspace with logical operators, used for sensing time-dependent $H(t)$
- Ladder codes ($n_p = 6$ example)



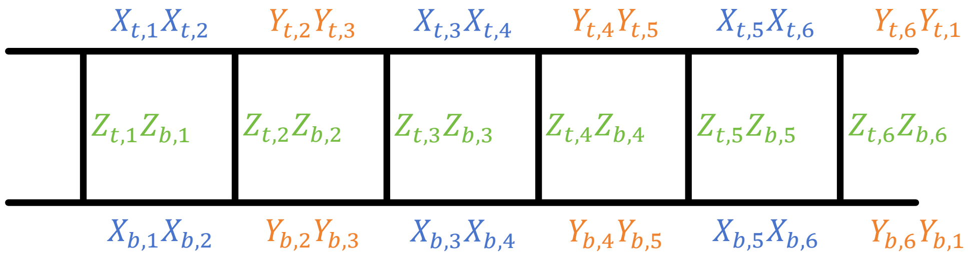
- Instantaneous stabilizer group: plaquette stabilizers and current checks

Round	0	1	2	3
Checks	ZZ	XX	ZZ	YY
Z_L after checks	$Z_{b,1}Z_{b,2}\cdots Z_{b,n_p}$	"	"	"
X_L after checks	$\pm X_{t,k}X_{b,k}$ or $\pm Y_{t,k}Y_{b,k}$	$\pm X_{t,k}X_{b,k}$	$\pm X_{t,k}X_{b,k}$ or $\pm Y_{t,k}Y_{b,k}$	$\pm Y_{t,k}Y_{b,k}$

The sign is determined by the history of check outcomes.

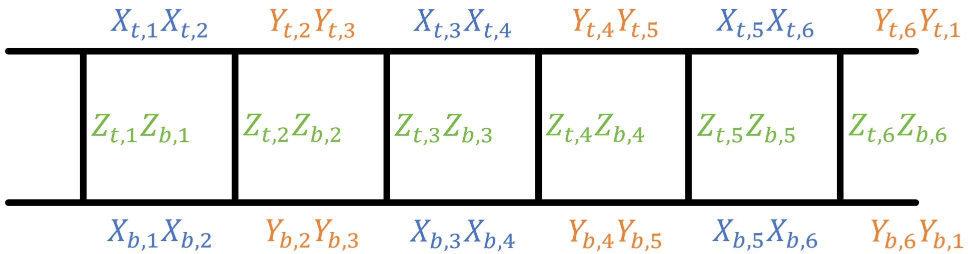
Hastings, Haah, Quantum 5, 564 (2021)

Error correction



- Check outcomes are random
- Syndrome bits: plaquette stabilizers computed from two recent rounds of checks
- Any flipped syndrome bit indicates errors
- A single check operator error (a Pauli error proportional to check operator) occurring at any rounds can be corrected
- Low rate check operator errors can be corrected (a logical error can only occur if at least $n_p/4$ check operator errors occur)

Floquet code metrology



- Hamiltonian $H^{(k)} = \begin{cases} \theta \sum_{i=1}^{n_p} X_{t,i}X_{b,i}\,, & k \text{ odd} \\ \theta \sum_{i=1}^{n_p} Y_{t,i}Y_{b,i}\,, & k \text{ even} \end{cases}$
- Assume low rate check operator noise right before checks

- After round of ZZ checks, apply unitary control to rotate the code space to be stabilized by $-ZZ$ checks.

Summary and outlook

- We demonstrate that subsystem QEC provides a new direction that substantially simplifies the metrological protocol.
- Our approach is particularly useful when measurement noise dominates.
- Heisenberg limit for a time-dependent metrology task can be achieved via Floquet codes.
- Can we allow imperfect QEC control and measurements?

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