k-Positivity and high-dimensional bound entanglement under symplectic group symmetries

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Outline

1. Entanglement, partial transpose, and Schmidt number

2. Symplectic group symmetry, k-positivity, and PPT entanglement

3. Further results & Open questions

Outline

Quantum entanglement

A bipartite state $\rho_{AB} \in \mathcal{D}(H_{AB})$ is called separable if

$$\rho_{AB} \in \mathcal{SEP} := \mathrm{conv}\{\sigma_A \otimes \sigma_B : \sigma_A \in \mathcal{D}(H_A),\ \sigma_B \in \mathcal{D}(H_B)\}.$$

Otherwise, ρ_{AB} is called entangled.

- Separable $\implies$ PPT (Positive Partial Transpose):

$$\rho_{AB} = \sum p_i\, \sigma_A^{(i)} \otimes \sigma_B^{(i)} \implies \rho_{AB}^{\Gamma} = \sum p_i\, \sigma_A^{(i)} \otimes (\sigma_B^{(i)})^{\top} \geq 0.$$

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- "PPT $\not\Rightarrow$ separable" unless $(d_A, d_B) = (2,2), (2,3)$, or $(3,2)$.
- (Gurvits 2003) Determining the membership in $\mathcal{SEP}$ is NP-hard.

Question: Can PPT (bound) entanglement be genuinely high-dim.?

Schmidt number: quantification of entanglement

Definition (Terhal and Horodecki 2000)

The Schmidt number of ρ_{AB} is defined as

$$SN(\rho_{AB}) := \min\{k : \rho_{AB} \in \mathcal{SN}_k\},$$

where $\mathcal{SN}_k := \mathrm{conv}\left\{|\psi\rangle\langle\psi| \;\middle|\; SR(|\psi\rangle) \leq k, \;\; \|\psi\|_2 = 1\right\}$.

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- $1 \leq SN(\rho_{AB}) \leq \min(d_A, d_B)$.
- ρ_{AB} is $\begin{cases} \text{separable iff } SN(\rho_{AB}) = 1, \\ \text{entangled iff } SN(\rho_{AB}) \geq 2. \end{cases}$

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- (Buscemi and Datta 2011) Schmidt number = one-shot (zero-error) entanglement cost = dimensionality of entanglement.
- There are very few examples whose Schmidt numbers are known.

Main question: Schmidt number of PPT states

Main question (Huber, Lancien, Lami, and Müller-Hermes 2018)

What is the maximum of $SN(\rho_{AB})$ given that $\rho_{AB} \in \mathcal{D}(H_{AB})$ is PPT?

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- (Chen, Yang, and Tang 2017) Construction of a $d \otimes d$ PPT state with $SN \gtrsim \log_2 d$.

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- (Huber, Lancien, Lami, and Müller-Hermes 2018) Construction of a $d \otimes d$ PPT state with $SN \geq \lfloor \frac{d-1}{4} \rfloor$.
- (Pál and Vértesi 2019; Cariello 2020) Construction of $d \otimes d$ PPT states with $SN = \lfloor \frac{d}{2} \rfloor$.
- (Krebs and Gachechiladze 2024; 2026) Construction of a $d \otimes d$ PPT state with $SN = \frac{d+1}{2}$ for $d = 5, 7, 9$.

Main result: overview

We consider a two-parameter family of bipartite quantum states

$$\rho_{a,b}^{(d)} := \frac{1-a-b}{d^2} I_{d^2} + a|\phi_d^+\rangle\langle\phi_d^+| + \frac{b}{d}(I_d \otimes \Omega_d) F_d (I_d \otimes \Omega_d)^\dagger \in M_d \otimes M_d,$$

where
$$\begin{cases} d : \text{even}, \\ |\phi_d^+\rangle := \frac{1}{\sqrt{d}} \sum_{i=1}^d |ii\rangle \in \mathbb{C}^d \otimes \mathbb{C}^d : \text{ max. entangled state}, \\ F_d := \sum_{i,j=1}^d |ij\rangle\langle ji| \in M_d \otimes M_d : \text{ flip operator}, \\ \Omega_d := \begin{pmatrix} 0 & I_{d/2} \\ -I_{d/2} & 0 \end{pmatrix} \in M_d. \end{cases}$$

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Example:

- $\rho_{a,0}^{(d)} = \frac{1-a}{d^2} I_{d^2} + a|\phi^+\rangle\langle\phi^+|$ are called the isotropic states.
- $(I_d \otimes \Omega_d)^\dagger \rho_{0,b}^{(d)} (I_d \otimes \Omega_d) = \frac{1-b}{d^2} I_{d^2} + \frac{b}{d} F_d$ are called the Werner states.

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Theorem (P. 2026+)

For every $d \geq 4$ even,

1. *The Schmidt number of $\rho_{a,b}^{(d)}$ is completely characterized.*
2. *There exists a (triangular) region $R \subset \mathbb{R}^2$ such that*

$$(a,b) \in R \iff \rho_{a,b}^{(d)} \text{ is a PPT state with Schmidt number } d/2.$$

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Remark:

- $(a,b) = (\frac{1}{d+2}, \frac{1}{d+2})$ corresponds to the example of [PV19].
- The family $\left\{\rho_{a,b}^{(d)}\right\}_{a,b}$ is obtained from the symplectic group symmetry.

Outline

Symplectic group symmetry

Let $d \geq 4$ be an even integer.

- A (complex) symplectic form $\langle \cdot, \cdot \rangle_\Omega : \mathbb{C}^d \times \mathbb{C}^d \to \mathbb{C}$ is defined by

$$\langle \mathbf{x}, \mathbf{y} \rangle_\Omega := \mathbf{x} \cdot \Omega_d \mathbf{y} = \sum_{i=1}^{d/2} (x_i y_{i+\frac{d}{2}} - x_{i+\frac{d}{2}} y_i), \quad \mathbf{x}, \mathbf{y} \in \mathbb{C}^d.$$

- (Unitary) Symplectic group

$$\begin{aligned} Sp(d) &:= \left\{ S \in \mathcal{U}_d \mid \langle S\mathbf{x}, S\mathbf{y} \rangle_\Omega = \langle \mathbf{x}, \mathbf{y} \rangle_\Omega \, \forall \, \mathbf{x}, \mathbf{y} \in \mathbb{C}^d \right\} \\ &= \left\{ S \in \mathcal{U}_d \mid S^\top \Omega_d S = \Omega_d \right\}. \end{aligned}$$

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Proposition (Schur-Weyl duality for the symplectic group)

A matrix $\rho \in M_d \otimes M_d$ satisfies the $(\overline{S} \otimes S)$-invariance

$$(\overline{S} \otimes S) \, \rho \, (\overline{S} \otimes S)^\dagger = \rho, \quad S \in Sp(d),$$

iff $\rho \in \mathrm{span}\{I_{d^2}, |\phi^+\rangle\langle\phi^+|, (I_d \otimes \Omega_d) F_d (I_d \otimes \Omega_d)^\dagger\}$. In particular,

$$\{\rho_{a,b}\}_{a,b} = \{(\overline{S} \otimes S)\text{-inv.} + \text{normalization}\}.$$

Duality on quantum entanglement

Recall: A linear map $\mathcal{L} : B(H_A) \to B(H_B)$ is called

$$\begin{cases} k\text{-positive if } \mathrm{id}_k \otimes \mathcal{L} : M_k \otimes B(H_A) \to M_k \otimes B(H_B) \text{ is positive,} \\ \text{decomposable if } \mathcal{L} = \Phi_1 + \top \circ \Phi_2 \text{ for some CP linear maps } \Phi_1, \Phi_2. \end{cases}$$

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Proposition (Størmer 1982; Terhal and Horodecki 2000)

1. *If $\rho = \rho_{AB}$ is a bipartite state, then*

$$\rho \text{ is PPT} \iff \mathrm{Tr}(\rho C_{\mathcal{L}}) \geq 0 \;\forall \mathcal{L} \in \mathcal{DEC} \text{ (decomposable)},$$
$$SN(\rho) \leq k \iff \mathrm{Tr}(\rho C_{\mathcal{L}}) \geq 0 \;\forall \mathcal{L} \in \mathcal{POS}_k \text{ (k-positive)}.$$

2. *TFAE:*
   1. $\exists$ *a* $d_A \otimes d_B$ *PPT state* ρ_{AB} *with* $SN(\rho_{AB}) > k$.
   2. $\exists$ *a k-positive map* $\mathcal{L} : M_{d_A} \to M_{d_B}$ *which is non-decomposable.*

Duality under group symmetry

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Proposition

1. *If $\rho = \rho_{AB}$ is an $(\overline{S} \otimes S)$-invariant bipartite state, then*

$$\rho \text{ is PPT} \iff \mathrm{Tr}(\rho C_{\mathcal{L}}) \geq 0 \ \ \forall \mathcal{L} \in \mathcal{C} \cap \mathcal{DEC},$$
$$SN(\rho) \leq k \iff \mathrm{Tr}(\rho C_{\mathcal{L}}) \geq 0 \ \ \forall \mathcal{L} \in \mathcal{C} \cap \mathcal{POS}_k,$$

   where $\mathcal{C} := \{\mathcal{L} : C_{\mathcal{L}} \text{ is } (\overline{S} \otimes S)\text{-inv.}\}$

2. *TFAE:*
   1. *$\exists$ a PPT state with $SN > k$ in $\mathrm{Inv}(\overline{S} \otimes S)$,*
   2. *$\exists$ a k-positive non-decomposable map in $\mathcal{C}$.*

k-positivity of symplectic covariant maps

Recall: $\rho_{a,b} = \frac{1-a-b}{d^2} I_{d^2} + a|\phi^+\rangle\langle\phi^+| + \frac{b}{d}(I_d \otimes \Omega_d)F_d(I_d \otimes \Omega_d)^\dagger$ is $\overline{S} \otimes S$-invariant. By taking $\mathcal{L}_{p,q}$ satisfying $C_{\mathcal{L}_{p,q}} = \rho_{p,q}$, i.e.,

$$\mathcal{L}_{p,q}(X) := (1-p-q)\frac{\mathrm{Tr}(X)}{d} I_d + pX + q\Omega_d X^T \Omega_d^\dagger$$

we have

$$SN(\rho_{a,b}) \leq k \iff \mathrm{Tr}(\rho_{a,b} C_{\mathcal{L}_{p,q}}) \geq 0 \quad \forall\ k\text{-positive } \mathcal{L}_{p,q}.$$

Question

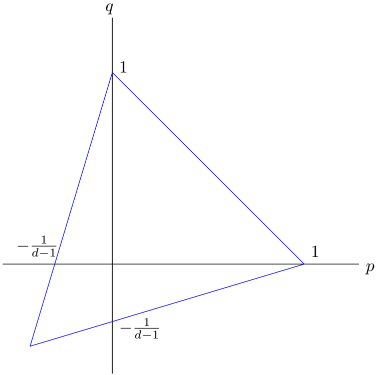
What is the maximal k such that the linear map $\mathcal{L}_{p,q}$ is k-positive and non-decomposable for some $(p,q) \in \mathbb{R}^2$?

Result 1: k-positivity of symplectic covariant maps

The k-positivity of $\mathcal{L}_{p,q}$ can be completely characterized.

Theorem (P. 2026+)

The linear map $\mathcal{L}_{p,q}(X) := (1-p-q)\frac{\mathrm{Tr}(X)}{d} I_d + pX + q\Omega_d X^T \Omega_d^\dagger$ *is*
(i) positive iff:



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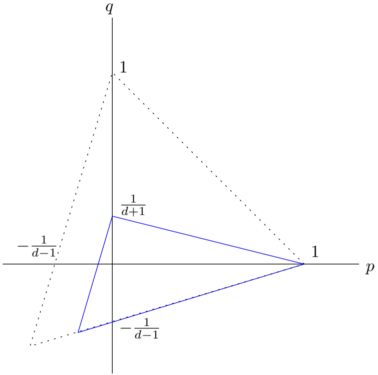

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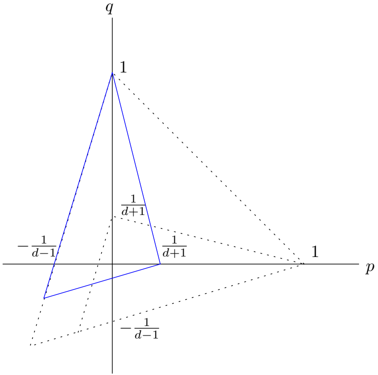

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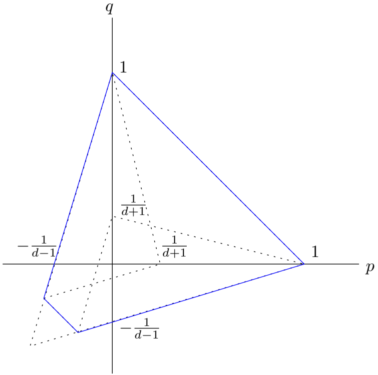

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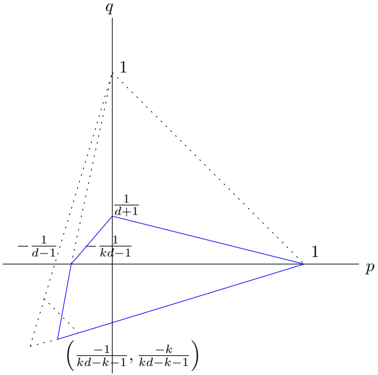

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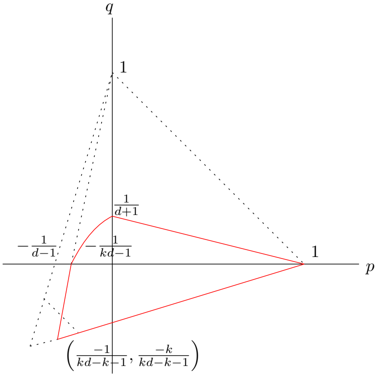

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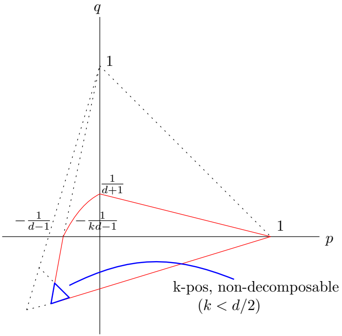
Corollary (P. 2026+)

The linear map $\mathcal{L}_{p,q}(X) := (1-p-q)\frac{\mathrm{Tr}(X)}{d} I_d + pX + q\Omega_d X^T \Omega_d^\dagger$ is k-positive and indecomposable $(k < d/2)$ iff:



$\longrightarrow$ The first construction of $(d/2-1)$-positive indecomposable maps!

Entanglement of Symplectic invariant states

Recall:

$$\rho_{a,b} = \tfrac{1-a-b}{d^2} I_{d^2} + a|\phi_d^+\rangle\langle\phi_d^+| + \tfrac{b}{d}(I_d \otimes \Omega_d)F_d(I_d \otimes \Omega_d)^\dagger = C_{\mathcal{L}_{a,b}}.$$

Since $\rho_{a,b}$ is $\overline{S} \otimes S$-invariant, we have

$$\begin{aligned} SN(\rho_{a,b}) \le k \iff & \mathrm{Tr}(C_{\mathcal{L}_{p,q}}\, \rho_{a,b}) = \mathrm{Tr}(\rho_{p,q}\, \rho_{a,b}) \ge 0 \quad \forall\, k\text{-positive } \mathcal{L}_{p,q} \\ \iff & (p \;\; q) \begin{pmatrix} d-1 & -1 \\ -1 & d-1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} \ge -\frac{1}{d+1} \quad \forall\, k\text{-positive } \mathcal{L}_{p,q}. \end{aligned}$$

Result 2: PPT states with high Schmidt number

The Schmidt number of $\rho_{a,b}$ can be completely characterized.

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Let $\rho_{a,b} := \frac{1-a-b}{d^2} I_{d^2} + a|\phi^+\rangle\langle\phi^+| + \frac{b}{d}(I_d \otimes \Omega_d)F_d(I_d \otimes \Omega_d)^\dagger$.

1. *If* $\rho_{a,b}$ *is PPT, then* $SN(\rho_{a,b}) \leq d/2$.

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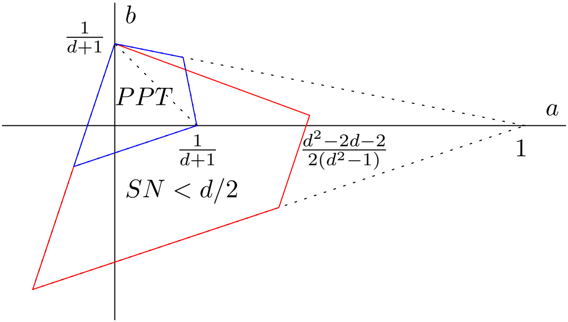

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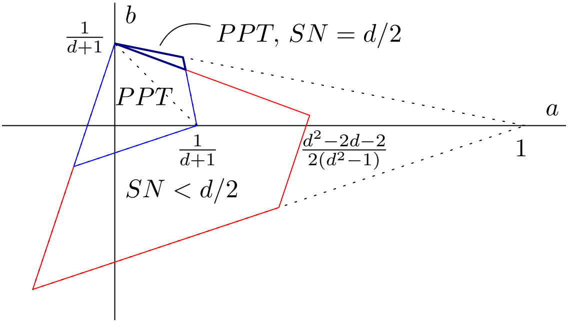
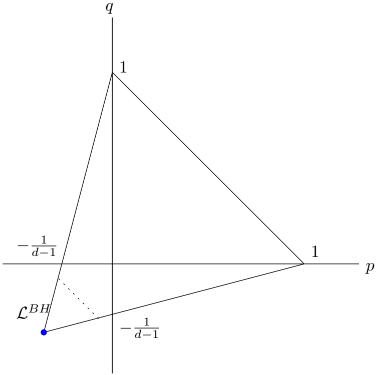


Outline

Breuer-Hall map

The Breuer-Hall map (Breuer 2006; Hall 2006) is defined as

$$\mathcal{L}^{\mathrm{BH}} := \mathcal{L}_{\frac{-1}{d-2},\frac{-1}{d-2}} : X \mapsto \frac{1}{d-2}(\mathrm{Tr}(X) I_d - X - \Omega_d X \Omega_d^{\dagger}).$$



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Theorem (Breuer 2006; Chruscinski and Kossakowski 2008)

1. $\mathcal{L}^{\mathrm{BH}}$ *is an optimal positive non-decomposable map, i.e., there is no other positive linear map* $\mathcal{L}'$ *such that,*

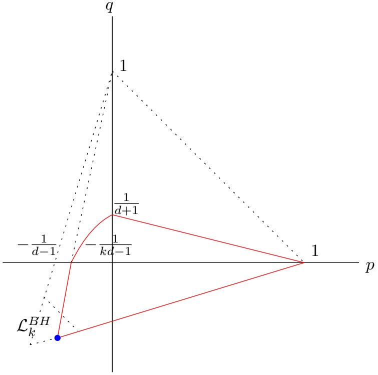
$$\{\rho \in \mathcal{PPT} : \mathrm{Tr}(\rho\, C_{\mathcal{L}^{\mathrm{BH}}}) < 0\} \subsetneq \{\rho \in \mathcal{PPT} : \mathrm{Tr}(\rho\, C_{\mathcal{L}'}) < 0\}.$$

2. $\mathcal{L}^{\mathrm{BH}}$ *is atomic, i.e., there exist no 2-positive linear maps* $\mathcal{L}_1$ *and* $\mathcal{L}_2$ *such that* $\mathcal{L}^{\mathrm{BH}} = \mathcal{L}_1 + \top \circ \mathcal{L}_2$.

Result: k-Breuer-Hall map

We can propose k-positive version of the Breuer-Hall-type map:

$$\mathcal{L}_k^{\mathrm{BH}}(X) := \frac{1}{kd-k-1}\left(k\,\mathrm{Tr}(X) I_d - X - k\,\Omega_d X\,\Omega_d^{\dagger}\right), \quad X \in M_d.$$



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Corollary (P. 2026+)

For $k = 1, 2, \ldots, d/2-1$,

1. *$\mathcal{L}_k^{\mathrm{BH}}$ is an optimal k-positive indecomposable map, i.e., there is no other k-positive linear map $\mathcal{L}'$ such that,*

$$\{\rho \in \mathcal{PPT} : \mathrm{Tr}(\rho\, C_{\mathcal{L}_k^{\mathrm{BH}}}) < 0\} \subsetneq \{\rho \in \mathcal{PPT} : \mathrm{Tr}(\rho\, C_{\mathcal{L}'}) < 0\}.$$

2. *There exist no $(k+1)$-positive linear maps $\mathcal{L}_1$ and $\mathcal{L}_2$ such that $\mathcal{L}_k^{\mathrm{BH}} = \mathcal{L}_1 + \top \circ \mathcal{L}_2$.*

Result: PPT^2 conjecture under symplectic group symmetry

A quantum channel Φ is called PPT (resp. entanglement-breaking) if C_Φ is PPT (resp. separable).

The PPT^2 conjecture (Christandl 2012)

$$\Phi_1 \text{ and } \Phi_2 \text{ are PPT} \implies \Phi_1 \circ \Phi_2 \text{ is entanglement-breaking ?}$$

Question: Does PPT $(\overline{S} \otimes S)$-invariant state $\rho_{a,b}$ with high Schmidt number provide a counterexample?

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Corollary (P. 2026+)

If $\Phi_1, \Phi_2 \in \mathcal{C} = \{\mathcal{L} : C_{\mathcal{L}} \text{ is } (\overline{S} \otimes S)\text{-inv.}\}$ are PPT, then $\Phi_1 \circ \Phi_2$ is EB.

Open: If $\Phi_1 \in \mathcal{C}$ is PPT, then is $\Phi_1 \circ \Phi_2$ EB for arbitrary PPT Φ_2?

Conclusion and outlook

In this work, we provided

- a two-parameter family of PPT states $\rho_{a,b}^{(d)}$ having $SN = d/2$,
- a two-parameter family of linear maps $\mathcal{L}_{p,q}^{(d)}$ which is $(d/2-1)$-positive and non-decomposable.

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Further questions:

- Can we improve the current Schmidt number (& k-positivity) bound by considering the extensions of symplectic group symmetry classes?
- Does the current characterization provide any other interesting consequences (e.g., distillability, extendibility, channel capacities)?

Thank you for your attention!