The Rate Functions of Quantum State Tomography

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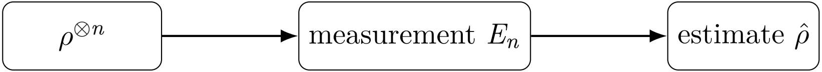
Quantum state tomography

Given n copies of an unknown state ρ, how do we design measurements so that we can estimate ρ?



n copies of the unknown state

Suppose tomography gives a *wrong* answer. How unlikely is a particular wrong estimate $\sigma \neq \rho$?

$$\Pr(\hat{\rho} \approx \sigma) \asymp \exp\big(-n\, I(\sigma \| \rho)\big), \quad n \to \infty.$$

The quantity $I(\sigma \| \rho)$ is the large deviation rate function of the tomography protocol.

Our contribution

1. We prove that among covariant (basis-independent) tomography protocols, the optimal rate is the reverse relative entropy (Keyl's rate):

$$\sup_{I \text{ is covariant}} I(\sigma\|\rho) = D_R(\sigma\|\rho).$$

   This resolves a conjecture of Keyl in 2006[1].

2. We prove that if covariance is dropped, the optimal rate becomes

$$\sup_{I} I(\sigma\|\rho) = D(\sigma\|\rho).$$

   Thus, noncovariant tomography can asymptotically approach relative entropy, resolving another conjecture of Keyl.

[1] Keyl, Michael. "Quantum state estimation and large deviations." Reviews in Mathematical Physics 18.01 (2006): 19-60.

Our contribution

3. **A new two-parameter family of tomography-induced divergences.**
   We construct a family of achievable rate functions

$$I_{\beta,\gamma}(\sigma\|\rho) = -(1+\gamma)\log\sum_{i=1}^{d}\left(a_i^{(\beta)}(\sigma,\rho)\,x_i^{\gamma-\beta}\right)^{\frac{1}{1+\gamma}},$$

   where

$$x = \mathrm{spec}^{\downarrow}(\sigma), \qquad a^{(\beta)}(\sigma,\rho) = \mathrm{spec}^{\downarrow}\big(\sigma^{\beta/2}\rho\sigma^{\beta/2}\big).$$

   In particular, $\beta=\gamma$ is the reverse sandwiched Rényi divergence.

4. **A quantum method of types.**
   Replacing Haar-random unitaries by unitary n-designs, we obtain sharp finite-n estimates for tomography. This gives a quantum analogue of the classical method of types.

Roadmap

1. Background
2. Keyl's Conjectures
3. Tomography-Induced Divergences
4. Quantum Method of Types

Preliminaries

Notation:

- $\mathcal{H}_n = \mathcal{H}^{\otimes n}$: Tensor product Hilbert space
- $\mathcal{D}_d$: Set of density operators
- $\mathcal{U}(d)$: Set of unitary operators
- Σ_d^n: Set of empirical types with denominator n

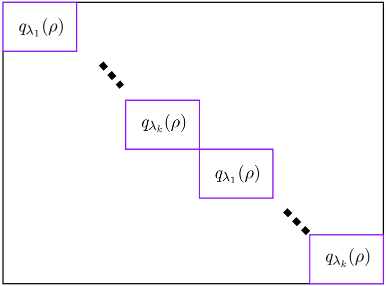
Schur-Weyl duality

$$(\mathbb{C}^d)^{\otimes n} = \bigoplus_{\lambda \vdash_d n} \mathcal{P}_\lambda \otimes \mathcal{Q}_\lambda$$

Schur-transform: U_{Schur}

$$U_{\text{Schur}} \rho^{\otimes n} U_{\text{Schur}}^\dagger = \bigoplus_{\lambda \vdash_d n} |\lambda\rangle\langle\lambda| \otimes \mathbf{1}_{\mathcal{P}_\lambda} \otimes q_\lambda(\rho)$$

Treat λ as a vector, and $\bar{\lambda} = \frac{\lambda}{n}$ as an empirical type



Divergences

- Quantum Relative Entropy:

$$D(\sigma \| \rho) = \mathrm{Tr}[\sigma(\log \sigma - \log \rho)]$$

- Reverse Relative Entropy/Keyl's rate: $\sigma = U \, \mathrm{diag}(x) \, U^\dagger$

$$D_R(\sigma \| \rho) = \mathrm{Tr}[\sigma \log \sigma] - \sum_{k=1}^{d} x_k \log \frac{\Delta_k(U^\dagger \rho U)}{\Delta_{k-1}(U^\dagger \rho U)}$$

, $\Delta_k(\cdot)$: determinant of leading principal $k \times k$ minor

  - $D(\sigma \| \rho) \geq D_R(\sigma \| \rho)$, and is equal when $[\sigma, \rho] = 0$
  - $D(\sigma \| \rho) > D_R(\sigma \| \rho)$ when $D_R(\sigma \| \rho) < \infty$ and $[\sigma, \rho] \neq 0$
  - D_R does not satisfy DPI for all channels

Quantum State Tomography

A tomography protocol, T_n, learns ρ by measuring $\rho^{\otimes n}$

- Described by a **POVM** $\{M_\sigma^{(n)}\}$ and **measure** μ_n

$$\int_{\mathcal{D}_d} M_\sigma^{(n)} \, d\mu_n(\sigma) = \mathbf{1}_{\mathcal{H}_n}$$

- Probability density:

$$P\left(\sigma|\rho\right) = \mathrm{Tr}\left[\rho^{\otimes n} M_\sigma^{(n)}\right]$$

- Induced measure: $A \subset \mathcal{D}_d$

$$\Pr\left(\mathsf{T}_n(\rho^{\otimes n}) \in A\right) \equiv \widehat{\mu}_n(A|\rho)$$

$$\widehat{\mu}_n(A|\rho) = \int_A P(\sigma|\rho) \, d\mu_n(\sigma), \quad \text{with} \quad \widehat{\mu}_n(\mathcal{D}_d|\cdot) = 1$$

Tomographic Large Deviations

The probability of seeing

- Large deviation rate function:

$$\lim_{n\to\infty} -\frac{1}{n} \log \widehat{\mu}_n(A|\rho) = \inf_{\sigma\in A} I(\sigma|\rho)$$

- *Consistent* protocols: For $\varepsilon > 0$

$$\lim_{n\to\infty} \widehat{\mu}_n\left(N_\varepsilon(\rho)|\rho\right) = 1, \qquad N_\varepsilon(\rho) := \{\tau \in \mathcal{D}_d : \|\tau - \rho\|_1 < \varepsilon\}$$

- *Covariant* protocols: $\forall U \in \mathcal{U}(d)$

$$M^{(n)}_{U\sigma U^\dagger} = U^{\otimes n} M^{(n)}_\sigma (U^\dagger)^{\otimes n}$$

  - Basis independent (i.e. no dependence on ρ)

Connection to Hypothesis Testing

For closed sets $A, B \subset \mathcal{D}_d$, given $\omega^{\otimes n}$

$$H_0 : \omega \in A$$
$$H_1 : \omega \in B$$

Given an effect $M^{(n)}$, the probability of type I and type II error is

$$\alpha(M^{(n)}) = \sup_{\sigma \in A} \mathrm{Tr}[(\mathbf{1} - M^{(n)})\sigma^{\otimes n}], \qquad \beta(M^{(n)}) = \sup_{\rho \in B} \mathrm{Tr}[M^{(n)}\rho^{\otimes n}]$$

A consistent tomography protocol yields

$$\alpha\left(M_{\mathrm{Tom}}^{(n)}\right) \to 0, \qquad \underline{\lim}_{n\to\infty} -\frac{1}{n}\log\beta\left(M_{\mathrm{Tom}}^{(n)}\right) \geq \inf_{\substack{\sigma\in A\\ \rho\in B}} I(\sigma|\rho)$$

Large deviation rate → Achievable Stein exponent

Roadmap

Keyl's Algorithm[2]

1. Measure $\rho^{\otimes n}$ in the Schur-basis
   - Get a Young diagram $\lambda \vdash_d n$, and reduced state $q_\lambda(\rho)$
2. Select $U \in \mathcal{U}(d)$ according to the Haar measure, μ_{Haar}
3. Rotate to $q_\lambda(U^\dagger \rho U)$, and project onto the highest weight component: $|\phi_\lambda\rangle\langle\phi_\lambda|$

In the Schur basis:

$$M^{(n)}_{U\,\mathrm{diag}(\bar{\lambda})\,U^\dagger} = \dim \mathcal{Q}_\lambda \cdot |\lambda\rangle\langle\lambda| \otimes \mathbf{1}_{\mathcal{P}_\lambda} \otimes \left(q_\lambda(U)\;|\phi_\lambda\rangle\langle\phi_\lambda|\;q_\lambda(U^\dagger)\right)$$

Properties:

- *Consistent* and *covariant*
- Rate function is $D_R(\sigma\|\rho)$

[2] Keyl, "Quantum state estimation and large deviations".

Keyl's Rate Function

- History of D_R
  - First introduced by Keyl
  - Limit $\alpha \to 1$ of reverse sandwiched Rényi divergence[3]
  - Rate of 'reverse' asymptotic pinching[4]
  - Stein exponent in composite HT problems[5]
- For $\sigma = \sum_{i=1}^{d} x_i(\sigma)|i\rangle\langle i|$, let $\Pi_k := \sum_{i=1}^{k} |i\rangle\langle i|$, then

$$0 \le D(\sigma\|\rho) - D_R(\sigma\|\rho)$$

$$= \sum_{k=1}^{d} (x_k(\sigma) - x_{k+1}(\sigma)) \left[\mathrm{Tr}\log\left(\Pi_k \rho \Pi_k\right) - \mathrm{Tr}\left(\Pi_k \log \rho \Pi_k\right)\right]$$

[3] Audenaert and Datta, "α-z-Rényi relative entropies".

[4] Lipka-Bartosik et al., "Quantum dichotomies and coherent thermodynamics beyond first-order asymptotics".

[5] Hayashi and Fang, *Operational interpretation of the reverse sandwiched Rényi divergences in composite quantum hypothesis testing.*

Keyl's Conjectures

Classes of rate functions:

- $\mathcal{E}$: Set of consistent rate functions
- $\mathcal{E}^c := \{I \in \mathcal{E} : I \text{ is covariant}\}$
- $\mathcal{E}^0 := \{I \in \mathcal{E} : I(\sigma|\rho) \text{ is lower semicontinuous in } \rho\}$

Keyl's conjectures[6]:

1. $\sup_{I \in \mathcal{E}} I(\sigma|\rho) = D(\sigma\|\rho)$
2. $\sup_{I \in \mathcal{E}^c} I(\sigma|\rho) = D_R(\sigma\|\rho)$
3. $\sup_{I \in \mathcal{E}^0} I(\sigma|\rho) = D_R(\sigma\|\rho)$

[6] Keyl, "Quantum state estimation and large deviations".

Keyl's First Conjecture

Theorem

For any consistent and covariant tomography protocol with rate function $I(\sigma|\rho)$*,*

$$I(\sigma|\rho) \leq D_R(\sigma\|\rho), \qquad \forall \sigma, \rho \in \mathcal{D}_d$$

- Covariant tomography protocols have 'nice' characterization in terms of Schur-basis and μ_{Haar}
- Consistent protocols are forced to assign mass around the highest weight component

Keyl's Second Conjecture

Informal Conjecture: For a fixed fixed pair σ, ρ and any $\varepsilon > 0$, there exists $I_{(\sigma,\rho),\varepsilon} \in \mathcal{E}$ so that

$$I_{(\sigma,\rho),\varepsilon}(\sigma|\rho) \geq D(\sigma\|\rho) - \varepsilon.$$

When $(\omega, \tau) \neq (\sigma, \rho)$, we allow

$$I_{(\sigma,\rho),\varepsilon}(\omega|\tau) \ll D(\omega\|\tau)$$

Theorem

For any $\sigma, \rho \in \mathcal{D}_d$,

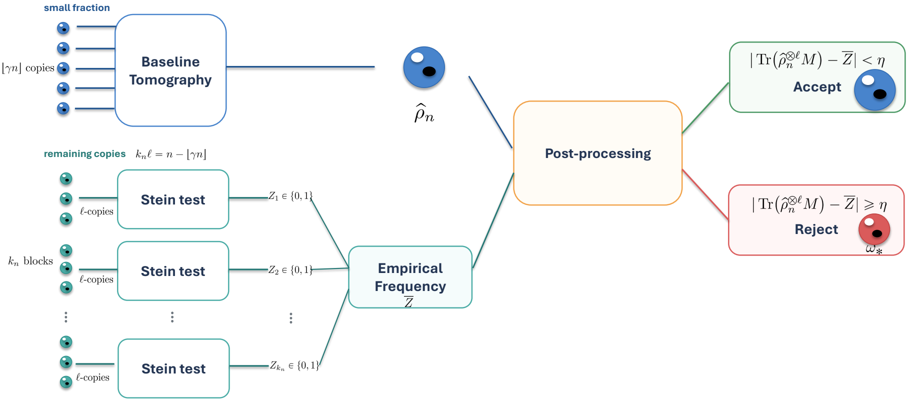
$$\sup_{I \in \mathcal{E}} I(\sigma|\rho) = D(\sigma\|\rho).$$

- Quantum Stein's lemma: if $I(\sigma|\rho)$ is consistent, then

$$I(\sigma|\rho) \leq D(\sigma\|\rho)$$

- We give a consistent protocol approaching $D(\sigma\|\rho)$ for a specific pair σ, ρ
  - * May perform poorly for other pairs

Protocol Diagram



Roadmap

Tomography generates a two-parameter family

For every $0 < \beta \leq \gamma$, we construct a tomography protocol whose large-deviation rate function is

Achievable rate function

$$I_{\beta,\gamma}(\sigma\|\rho) = -(1+\gamma)\log\sum_{i=1}^{d}\left(a_i^{(\beta)}(\sigma,\rho)\,x_i^{\gamma-\beta}\right)^{\frac{1}{1+\gamma}}.$$

$$x = (x_1,\ldots,x_d) = \underbrace{\mathrm{spec}^{\downarrow}(\sigma)}_{\substack{\text{eigenvalues of }\sigma\\ \text{in nonincreasing order}}} \qquad a^{(\beta)}(\sigma,\rho) = \mathrm{spec}^{\downarrow}\left(\sigma^{\beta/2}\rho\sigma^{\beta/2}\right).$$

Its i-th component is $a_i^{(\beta)}(\sigma,\rho)$.

Known protocols appear as special cases

Diagonal family: $\gamma = \beta$

$I_{\beta,\beta}$ is the reverse sandwiched Rényi divergence of order $\frac{\beta}{1+\beta}$ achieved by the order-β pretty-good measurement[a].

$$\underbrace{I_{\beta,\beta}(\sigma\|\rho)}_{\text{order-}\beta\text{ PGM}} \xrightarrow{\beta\to\infty} \underbrace{D_R(\sigma\|\rho)}_{\text{Keyl's protocol}}$$

[a] Hausladen and Wootters, A "pretty good" measurement for distinguishing quantum states, J. Mod. Opt. 41 (1994).

Vertical limit: $\gamma \to \infty$

For each fixed $\beta > 0$, define

$$I_{\beta,\infty}(\sigma\|\rho) := \lim_{\gamma\to\infty} I_{\beta,\gamma}(\sigma\|\rho).$$

This rate is achieved by the order-β tomography protocol of Haah et al [a].

[a] Haah et al, Sample-optimal tomography of quantum states, 2017

Roadmap

Unitary n-designs

$\mathcal{V}_n \subset \mathcal{U}(d)$ is a *unitary n-design* if $|\mathcal{V}_n| < \infty$, and $\forall A \in \mathrm{Herm}(\mathcal{H}^{\otimes \mathrm{n}})$:

$$\frac{1}{|\mathcal{V}_n|} \sum_{V \in \mathcal{V}_n} V^{\otimes n} A \, V^{\dagger \otimes n} = \int_{\mathcal{U}(d)} d\mu_{\mathrm{Haar}}(U) \; U^{\otimes n} A \, U^{\dagger \otimes n},$$

- Discretize the Haar integral

Properties:

- Unitary n-designs become dense in $\mathcal{U}(d)$ as $n \to \infty$
- Unitary n-designs with $|\mathcal{V}_n| = O(n^{4d^2})$ exist[7][8]

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[7] Kane, "Small designs for path-connected spaces and path-connected homogeneous spaces".

[8] Etayo, Marzo, and Ortega-Cerda, "Asymptotically optimal designs on compact algebraic manifolds".

Quantum Method of Types

Idea: Replace Haar integral in Keyl's protocol with unitary n-design

$$\hat{\Sigma}_n = \left\{ U \,\mathrm{diag}(\bar{\lambda})\, U^\dagger : U \in \mathcal{V}_n, \lambda \vdash_d n \right\}$$

In the Schur-basis:

$$M^{(n)}_{U\,\mathrm{diag}(\bar{\lambda})\,U^\dagger} = \frac{\dim \mathcal{Q}_\lambda \dim \mathcal{P}_\lambda}{|\mathcal{V}_n|} |\lambda\rangle\langle\lambda| \otimes \mathbf{1}_{\mathcal{P}_\lambda} \otimes q_\lambda(U^\dagger)|\phi_\lambda\rangle\langle\phi_\lambda| q_\lambda(U)$$

$$\mu_n(\sigma) = \mathbb{1}\{\sigma \in \hat{\Sigma}_n\}$$

Properties:

- $\left|\hat{\Sigma}_n\right| = O(n^{5d^2})$
- $\hat{\Sigma}_n$ becomes dense in $\mathcal{D}_d$
- For $\sigma_n \in \hat{\Sigma}_n$:

$$\frac{1}{\mathrm{poly}(n)} e^{-nD_R(\sigma_n \| \rho)} \leq P\left(\sigma_n | \rho\right) \leq \mathrm{poly}(n) e^{-nD_R(\sigma_n \| \rho)}$$

Conclusions

- Tomography is a quantum version of the empirical distribution
  - Tomography gives a universally *achievable* Stein exponent for composite quantum hypothesis testing
- $D_R(\sigma\|\rho)$ is best basis independent tomographic rate
- Combining unitary designs and Keyl's protocol gives us a quantum method of types

Future work:

- Determine which composite problems have D_R as optimal Stein exponent
- Information geometry of D_R
- Second order asymptotics of hypothesis testing using tomography

Thanks for listening!

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